Support Vector Machines: Training with Stochastic Gradient Descent

Machine Learning
Fall 2022

The slides are mainly from Vivek Srikumar
Support vector machines

- Training by maximizing margin
- The SVM objective
- Solving the SVM optimization problem
- Support vectors, duals and kernels
SVM objective function

\[
\min_{w,b} \frac{1}{2} w^T w + C \sum \max (0, 1 - y_i (w^T x_i + b))
\]

**Regularization term:**
- Maximize the margin
- Imposes a preference over the hypothesis space and pushes for better generalization
- Can be replaced with other regularization terms which impose other preferences

**Empirical Loss:**
- Hinge loss
- Penalizes weight vectors that make mistakes
- Can be replaced with other loss functions which impose other preferences

A hyper-parameter that controls the tradeoff between a large margin and a small hinge-loss
Outline: Training SVM by optimization

1. Review of convex functions and gradient descent
2. Stochastic gradient descent
3. Gradient descent vs stochastic gradient descent
4. Sub-derivatives of the hinge loss
5. Stochastic sub-gradient descent for SVM
6. Comparison to perceptron
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Solving the SVM optimization problem

\[
\min_{w, b} \quad \frac{1}{2} w^\top w + C \sum_i \max(0, 1 - y_i (w^\top x_i + b))
\]

This function is **convex** in \(w, b\)

For convenience, use simplified notation:

\[
\begin{align*}
    w_0 &\leftarrow w \\
    w &\leftarrow [w_0, b] \\
    x_i &\leftarrow [x_i, 1]
\end{align*}
\]

\[
\min_{w} \quad \frac{1}{2} w_0^\top w_0 + C \sum_i \max(0, 1 - y_i w^\top x_i)
\]
Recall: Convex functions

A function $f$ is **convex** if for every $u, v$ in the domain, and for every $\lambda \in [0,1]$ we have

$$f(\lambda u + (1 - \lambda)v) \leq \lambda f(u) + (1 - \lambda)f(v)$$
Recall: Convex functions

A function \( f \) is **convex** if for every \( u, v \) in the domain, and for every \( \lambda \in [0,1] \) we have

\[
f(\lambda u + (1 - \lambda)v) \leq \lambda f(u) + (1 - \lambda)f(v)
\]

From geometric perspective

Every tangent plane lies below the function

\[
f(x) \geq f(u) + \nabla f(u)^\top (x - u)
\]
Convex functions

\[ f(x) = -x \]
Linear functions

\[ f(x) = x^2 \]

\[ f(x) = \max(0, x) \]
\textit{max} is convex

Some ways to show that a function is convex:

1. Using the definition of convexity
2. Showing that the second derivative is nonnegative (for one dimensional functions)
3. Showing that the second derivative is positive semi-definite (for vector functions)
Not all functions are convex

These are concave

\[ f(\lambda u + (1 - \lambda)v) \geq \lambda f(u) + (1 - \lambda)f(v) \]

These are neither
Convex functions are convenient

A function \( f \) is **convex** if for every \( u, v \) in the domain, and for every \( \lambda \in [0,1] \) we have

\[
f(\lambda u + (1 - \lambda)v) \leq \lambda f(u) + (1 - \lambda)f(v)
\]

In general: Necessary condition for \( x \) to be a minimum for the function \( f \) is \( \nabla f(x) = 0 \)
Convex functions are convenient

A function $f$ is **convex** if for every $u, v$ in the domain, and for every $\lambda \in [0,1]$ we have

$$f(\lambda u + (1 - \lambda)v) \leq \lambda f(u) + (1 - \lambda)f(v)$$

In general: Necessary condition for $x$ to be a minimum for the function $f$ is $\nabla f(x) = 0$

For convex functions, this is both necessary *and* sufficient
Solving the SVM optimization problem

\[
\min_w \frac{1}{2} w_0^T w_0 + C \sum_i \max(0, 1 - y_i w^T x_i)
\]

This function is convex in \( w \)

- This is a quadratic optimization problem because the objective is quadratic

- Older methods: Used techniques from Quadratic Programming
  - Very slow

- No constraints, can use \textit{gradient descent}
  - Still very slow!
Gradient descent

General strategy for minimizing a function $J(w)$

- Start with an initial guess for $w$, say $w^0$
- Iterate till convergence:
  - Compute the gradient of $J$ at $w^t$
  - Update $w^t$ to get $w^{t+1}$ by taking a step in the opposite direction of the gradient

Intuition: The gradient is the direction of steepest increase in the function. To get to the minimum, go in the opposite direction
Gradient descent

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We are trying to minimize

$$J(w) = \frac{1}{2}w_0^T w_0 + C \sum_i \max(0, 1 - y_i w^T x_i)$$

**Intuition**: The gradient is the direction of steepest increase in the function. To get to the minimum, go in the opposite direction.
Gradient descent

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**Intuition**: The gradient is the direction of steepest increase in the function. To get to the minimum, go in the opposite direction
Gradient descent for SVM

1. Initialize $w^0$

2. For $t = 0, 1, 2, ...$
   1. Compute gradient of $J(w)$ at $w^t$. Call it $\nabla J(w^t)$

   
   2. Update $w$ as follows:

   $$w^{t+1} = w^t - r \nabla J(w^t)$$

   $r$: Called the learning rate.
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Gradient descent for SVM

1. Initialize \( \mathbf{w}^0 \)

2. For \( t = 0, 1, 2, \ldots \)
   
   1. Compute gradient of \( J(\mathbf{w}) \) at \( \mathbf{w}^t \). Call it \( \nabla J(\mathbf{w}^t) \)

We are trying to minimize

\[
J(\mathbf{w}) = \frac{1}{2} \mathbf{w}^\top \mathbf{w} + C \sum_i \max(0, 1 - y_i \mathbf{w}^\top \mathbf{x}_i)
\]

Gradient of the SVM objective requires summing over the entire training set

**Slow, does not really scale**

\( \eta \): Called the learning rate
Stochastic gradient descent for SVM

Given a training set $S = \{(x_i, y_i)\}$, $x \in \mathbb{R}^n$, $y \in \{-1, 1\}$

1. Initialize $w^0 = 0 \in \mathbb{R}^n$
2. For epoch = 1 ... T:

$J(w) = \frac{1}{2} w_0^T w_0 + C \sum_i \max(0, 1 - y_i w^T x_i)$

3. Return final $w$
Stochastic gradient descent for SVM

Given a training set $S = \{(x_i, y_i)\}$, $x \in \mathbb{R}^n$, $y \in \{-1, 1\}$

1. Initialize $w^0 = 0 \in \mathbb{R}^n$

2. For epoch = 1 ... $T$:
   1. Pick a random example $(x_i, y_i)$ from the training set $S$

3. Return final $w$
Stochastic gradient descent for SVM

Given a training set $S = \{(x_i, y_i)\}, \; x \in \mathbb{R}^n, \; y \in \{-1, 1\}$

1. Initialize $w^0 = 0 \in \mathbb{R}^n$

2. For epoch = 1 ... T:
   1. Pick a random example $(x_i, y_i)$ from the training set $S$
   2. Repeat $(x_i, y_i)$ to make a full dataset and take the derivative of the SVM objective at the current $w$ to be $\nabla J^t(w)$

3. Return final $w$
Given a training set $S = \{(x_i, y_i)\}$, $x \in \mathbb{R}^n$, $y \in \{-1, 1\}$

1. Initialize $w^0 = 0 \in \mathbb{R}^n$
2. For epoch $= 1 \ldots T$:
   1. Pick a random example $(x_i, y_i)$ from the training set $S$
   2. Repeat $(x_i, y_i)$ to make a full dataset and take the derivative of the SVM objective at the current $w$ to be $\nabla J^t(w)$
      
      $$J^t(w) = \frac{1}{2}w_0^T w_0 + C \cdot N \max(0, 1 - y_i w^T x_i)$$

3. Return final $w$
Stochastic gradient descent for SVM

Given a training set \( S = \{(x_i, y_i)\}, x \in \mathbb{R}^n, y \in \{-1, 1\} \)

1. Initialize \( w^0 = 0 \in \mathbb{R}^n \)
2. For epoch = 1 \( \ldots \) T:
   1. Pick a random example \((x_i, y_i)\) from the training set \( S \)
   2. Repeat \((x_i, y_i)\) to make a full dataset and take the derivative of the SVM objective at the current \( w \) to be \( \nabla J_t(w) \)

\[
J_t(w) = \frac{1}{2} w_0^T w_0 + C \sum_i \max(0, 1 - y_i w^T x_i)
\]

3. Return final \( w \)
Stochastic gradient descent in general

\[ f(w) = R(w) + C \sum_{n=1}^{N} l(x_n, w) \quad \Rightarrow \quad f(w) = R(w) + C \frac{1}{N} \sum_{n=1}^{N} N \cdot l(x_n, w) \]

\[ f(w) = R(w) + C \mathbb{E}_{p(k)}[N \cdot l(x_k, w)] \quad p(k) = \frac{1}{N} \quad (k = 1 \ldots N) \]

\[ f(w) = \mathbb{E}_{p(k)}[R(w) + C \cdot N \cdot l(x_k, w)] \]

\[ \nabla f(w) = \nabla \mathbb{E}_{p(k)}[R(w) + C \cdot N \cdot l(x_k, w)] \]

\[ = \mathbb{E}_{p(k)}[\nabla (R(w) + C \cdot N \cdot l(x_k, w))] \]

Unbiased stochastic gradient: ensure convergence

Randomly sample, duplicate it to make a full data, then compute the gradient
Stochastic gradient descent for SVM

Given a training set $S = \{(x_i, y_i)\}$, $x \in \mathbb{R}^n$, $y \in \{-1, 1\}$

1. Initialize $w^0 = 0 \in \mathbb{R}^n$
2. For epoch $= 1 \ldots T$:
   1. Pick a random example $(x_i, y_i)$ from the training set $S$
   2. Repeat $(x_i, y_i)$ to make a full dataset and take the derivative of the SVM objective at the current $w$ to be $\nabla J^t(w)$
      \[
      J^t(w) = \frac{1}{2} w_0^T w_0 + C \cdot \underset{i}{\max}(0, 1 - y_i w^T x_i)
      \]
   3. Update: $w \leftarrow w - \gamma_t \nabla J^t(w)$
3. Return final $w$
Stochastic gradient descent for SVM

Given a training set \( S = \{(x_i, y_i)\}, x \in \mathbb{R}^n, y \in \{-1, 1\} \)

1. Initialize \( w^0 = 0 \in \mathbb{R}^n \)
2. For epoch = 1 ... T:
   1. Pick a random example \((x_i, y_i)\) from the training set \( S \)
   2. Repeat \((x_i, y_i)\) to make a full dataset and take the derivative of the SVM objective at the current \( w \) to be \( \nabla J_t(w) \)
   3. Update: \( w \leftarrow w - \gamma_t \nabla J_t(w) \)
3. Return final \( w \)

This algorithm is guaranteed to converge to the minimum of \( J \) if \( \gamma_t \) is small enough.
Outline: Training SVM by optimization

✓ Review of convex functions and gradient descent
✓ Stochastic gradient descent
3. **Gradient descent vs stochastic gradient descent**
4. Sub-derivatives of the hinge loss
5. Stochastic sub-gradient descent for SVM
6. Comparison to perceptron
Gradient Descent vs SGD

Gradient descent
Gradient Descent vs SGD

Stochastic Gradient descent
Gradient Descent vs SGD

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Many more updates than gradient descent, but each individual update is less computationally expensive.
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Stochastic gradient descent for SVM

Given a training set \( S = \{(x_i, y_i)\}, \ x \in \mathbb{R}^n, \ y \in \{-1, 1\} \)

1. Initialize \( w^0 = 0 \in \mathbb{R}^n \)
2. For epoch = 1 … T:
   1. Pick a random example \( (x_i, y_i) \) from the training set \( S \)
   2. Treat \( (x_i, y_i) \) as a full dataset and take the derivative of the SVM objective at the current \( w \) to be \( \nabla J_t(w) \)
   3. Update: \( w \leftarrow w - \gamma_t \nabla J_t(w) \)
3. Return final \( w \)

What is the derivative of the hinge loss with respect to \( w \)? (The hinge loss is not a differentiable function!)

\[
J(w) = \frac{1}{2} w^0 w_0 + C \sum_i \max(0, 1 - y_i w^\top x_i)
\]
Hinge loss is **not** differentiable!

What is the derivative of the hinge loss with respect to $w$?

$$J^t(w) = \frac{1}{2} w_0^\top w_0 + C \cdot N \max(0, 1 - y_i w^\top x_i)$$
Detour: Sub-gradients

Generalization of gradients to non-differentiable functions
(Remember that every tangent lies below the function for convex functions)

Informally, a sub-tangent line at a point is any line that crosses the point and lies below the entire function.
A sub-gradient is the slope of the sub-tangent line
Sub-gradients

Formally, \( g \) is a subgradient to \( f \) at \( x \) if

\[
f(y) \geq f(x) + g^T(y - x) \quad \text{for all } y
\]
Sub-gradients

Formally, $g$ is a subgradient to $f$ at $x$ if

$$f(y) \geq f(x) + g^T(y - x) \quad \text{for all } y$$

$f$ is differentiable at $x_1$
Tangent at this point

$$f(x_1) + g_1^T(x - x_1)$$

$g_1$ is a gradient at $x_1$
Sub-gradients

Formally, \( g \) is a subgradient to \( f \) at \( x \) if

\[
f(y) \geq f(x) + g^T(y - x) \quad \text{for all } y
\]

\( f \) is differentiable at \( x_1 \)

Tangent at this point

\[
f(x_1) + g_1^T(x - x_1)
\]

\( g_1 \) is a gradient at \( x_1 \)

\[
f(x_2) + g_2^T(x - x_2)
\]

\( g_2 \) and \( g_3 \) are both subgradients at \( x_2 \)

[Example from Boyd]
Sub-gradient of the SVM objective

\[ J^t(w) = \frac{1}{2} w_0^T w_0 + C \cdot N \max(0, 1 - y_i w^T x_i) \]

**General strategy**: First solve the max and compute the gradient for each case.
Sub-gradient of the SVM objective

\[ J^t(w) = \frac{1}{2} w_0^\top w_0 + C \cdot N \max(0, 1 - y_i w^\top x_i) \]

**General strategy:** First solve the max and compute the gradient for each case

\[ \nabla J^t = \begin{cases} 
[w_0; 0] & \text{if } \max(0, 1 - y_i w^\top x_i) = 0 \\
[w_0; 0] - C \cdot N y_i x_i & \text{otherwise}
\end{cases} \]
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Stochastic sub-gradient descent for SVM

\[ \nabla J^t = \begin{cases} 
[w_0; 0] & \text{if } \max(0, 1 - y_i w^\top x_i) = 0 \\
[w_0; 0] - C \cdot N y_i x_i & \text{otherwise}
\end{cases} \]

Given a training set \( S = \{(x_i, y_i)\}, \ x \in \mathbb{R}^n, \ y \in \{-1,1\} \)

1. Initialize \( w^0 = 0 \in \mathbb{R}^n \)

3. Return \( w \)
Stochastic sub-gradient descent for SVM

\[ \nabla J^t = \begin{cases} 
[w_0; 0] & \text{if } \max(0, 1 - y_i w^\top x_i) = 0 \\
[w_0; 0] - C \cdot N y_i x_i & \text{otherwise}
\end{cases} \]

Given a training set \( S = \{(x_i, y_i)\}, \ x \in \mathbb{R}^n, y \in \{-1,1\} \)

1. Initialize \( w^0 = 0 \in \mathbb{R}^n \)
2. For epoch = 1 ... T:

3. Return \( w \)
Stochastic sub-gradient descent for SVM

\[ \nabla J_t = \begin{cases} [w_0; 0] & \text{if } \max(0, 1 - y_i w^T x_i) = 0 \\ [w_0; 0] - C \cdot N y_i x_i & \text{otherwise} \end{cases} \]

Given a training set \( S = \{(x_i, y_i)\}, \ x \in \mathbb{R}^n, \ y \in \{-1, 1\} \)

1. Initialize \( w^0 = 0 \in \mathbb{R}^n \)

2. For epoch = 1 ... T:
   1. For each training example \( (x_i, y_i) \in S \):
      
      Update \( w \leftarrow w - \gamma_t \nabla J_t \)

3. Return \( w \)
Stochastic sub-gradient descent for SVM

\[ \nabla J^t = \begin{cases} 
[w_0; 0] & \text{if } \max(0, 1 - y_i w^T x_i) = 0 \\
[w_0; 0] - C \cdot N y_i x_i & \text{otherwise}
\end{cases} \]

Given a training set \( S = \{(x_i, y_i)\}, x \in \mathbb{R}^n, y \in \{-1, 1\} \)

1. Initialize \( w^0 = 0 \in \mathbb{R}^n \)
2. For epoch = 1 … T:
   1. For each training example \((x_i, y_i) \in S:\)
      If \( y_i w^T x_i \leq 1, \)
      \[
      w \leftarrow w - \gamma_t [w_0; 0] + \gamma_t C N y_i x_i
      \]
      else
      \[
      w_0 \leftarrow (1 - \gamma_t) w_0
      \]
3. Return \( w \)
Stochastic sub-gradient descent for SVM

Given a training set $S = \{(x_i, y_i)\}$, $x \in \mathbb{R}^n$, $y \in \{-1,1\}$

1. Initialize $w^0 = 0 \in \mathbb{R}^n$

2. For epoch = 1 ... T:
   1. For each training example $(x_i, y_i) \in S$:
      
      If $y_i x_i^T w \leq 1$,
      
      $$w \leftarrow w - \gamma_t [w_0; 0] + \gamma_t C N y_i x_i$$
      
      else
      
      $$w_0 \leftarrow (1 - \gamma_t) w_0$$

3. Return $w$
Stochastic sub-gradient descent for SVM

Given a training set \( S = \{(x_i, y_i)\}, \quad x \in \mathbb{R}^n, \ y \in \{-1,1\} \)

1. Initialize \( w^0 = 0 \in \mathbb{R}^n \)

2. For epoch = 1 \( \ldots \) T:
   
   1. Shuffle the training set
   
   2. For each training example \( (x_i, y_i) \in S \):
      
      If \( y_i w^\top x_i \leq 1 \),
      
      \[
      w \leftarrow w - \gamma_t [w_0; 0] + \gamma_t C N y_i x_i
      \]
      
      else
      
      \[
      w_0 \leftarrow (1 - \gamma_t) w_0
      \]

3. Return \( w \)
Convergence and learning rates

With enough iterations, it will converge in expectation

Provided the step sizes are "square summable, but not summable"

- Step sizes $\gamma_t$ are positive
- Sum of squares of step sizes over $t = 1$ to $\infty$ is not infinite
- Sum of step sizes over $t = 1$ to $\infty$ is infinity

- Some examples: $\gamma_t = \frac{\gamma_0}{1+\frac{\gamma_0 t}{a}}$ or $\gamma_t = \frac{\gamma_0}{1+t}$
Convergence and learning rates

• Number of iterations to get to accuracy within $\epsilon$

• For strongly convex functions, $N$ examples, $d$ dimensional:
  – Gradient descent: $O(Nd \ln(1/\epsilon))$
  – Stochastic gradient descent: $O(d/\epsilon)$

• More subtleties involved, but SGD is generally preferable when the data size is huge
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Stochastic sub-gradient descent for SVM

Given a training set \( S = \{(x_i, y_i)\}, \ x \in \mathbb{R}^n, \ y \in \{-1,1\} \)

1. Initialize \( w^0 = 0 \in \mathbb{R}^n \)

2. For epoch = 1 ... T:
   1. Shuffle the training set
   2. For each training example \((x_i, y_i)\) ∈ S:
      - If \( y_i w^T x_i \leq 1 \),
        \[ w \leftarrow w - \gamma_t [w_0; 0] + \gamma_t C N y_i x_i \]
      - else
        \[ w_0 \leftarrow (1 - \gamma_t) w_0 \]

3. Return \( w \)

Compare with the Perceptron update:
If \( y_i w^T x_i \leq 0 \), update \( w \leftarrow w + r y_i x_i \)
Perceptron vs. SVM

• Perceptron: Stochastic sub-gradient descent for a different loss
  – No regularization though

\[ L_{\text{Perceptron}}(y, x, w) = \max(0, -yw^T x) \]

• SVM optimizes the hinge loss
  – With regularization

\[ L_{\text{Hinge}}(y, x, w) = \max(0, 1 - yw^T x) \]
SVM summary from optimization perspective

• Minimize regularized hinge loss

• Solve using stochastic sub-gradient descent
  – Very fast, run time does not depend on number of examples
  
  – Compare with Perceptron algorithm: similar framework with different objectives!
  
  – Compare with Perceptron algorithm: Perceptron does not maximize margin width
    • Perceptron variants can force a margin

• Other successful optimization algorithms exist
  – Eg: Dual coordinate descent, implemented in liblinear

Questions?