We often want to generate sets of random or pseudorandom points on the unit square for applications such as distribution ray tracing. There are several methods for doing this, such as jittering and Poisson disk sampling. These methods give us a set of $N$ reasonably equidistributed points on the unit square: $(u_1, v_1)$ through $(u_N, v_N)$.

Sometimes, our sampling space may not be square (e.g., a circular lens) or may not be uniform (e.g., a filter function centered on a pixel). It would be nice if we could write a mathematical transformation that would take our equidistributed points $(u_i, v_i)$ as input, and output a set of points in our desired sampling space with our desired density. For example, to sample a camera lens, the transformation would take $(u_i, v_i)$ and output $(r_i, \theta_i)$ such that the new points were approximately equidistributed on the disk of the lens.

It turns out that such transformation functions are well known in the field of Monte Carlo integration. A table of several transformations useful for computer graphics is given in Table I. The method for generating such transformations is discussed for the rest of this article. Note that several of these transformations can be simplified for simple densities. For example, to generate directions with a cosine distribution, use the Phong density with $n = 1$. To generate points on the unit hemisphere, use the sector on the unit sphere density with $\theta_1 = 0$, $\theta_2 = \pi/2$, $\phi_1 = 0$, and $\phi_2 = \pi$.

For Monte Carlo methods we must often generate random points according to some probability density function, or random rays according to a directional probability density. In this section a method for one and two dimensional random variables is described. The discussion closely follows that of Shreider (1966).
Table 1. Some Useful Transformations.

<table>
<thead>
<tr>
<th>Target space</th>
<th>Density</th>
<th>Domain</th>
<th>Transformation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radius $R$ disk</td>
<td>$p(r, \theta) = \frac{1}{\pi R^2}$</td>
<td>$\theta \in [0, 2\pi]$</td>
<td>$\theta = 2\pi u$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$r \in [0, R]$</td>
<td>$r = R/\sqrt{u}$</td>
</tr>
<tr>
<td>Sector of radius $R$ disk</td>
<td>$p(r, \theta) = \frac{2}{(\theta_2 - \theta_1)(r_2^2 - r_1^2)}$</td>
<td>$\theta \in [\theta_1, \theta_2]$</td>
<td>$\theta = \theta_1 + u(\theta_2 - \theta_1)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$r \in [r_1, r_2]$</td>
<td>$r = \sqrt{r_1^2 + u(r_2^2 - r_1^2)}$</td>
</tr>
<tr>
<td>Phong density exponent $n$</td>
<td>$p(\theta, \phi) = \frac{n + 1}{2\pi} \cos^n \theta$</td>
<td>$\theta \in \left[0, \frac{\pi}{2}\right]$</td>
<td>$\theta = \arccos\left((1 - u)^{1/n}\right)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\phi \in [0, 2\pi]$</td>
<td>$\phi = 2\pi v$</td>
</tr>
<tr>
<td>Separated triangle filter</td>
<td>$p(x, y)(1 -</td>
<td>x</td>
<td>X(1 -</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y \in [-1, 1]$</td>
<td>$y = \begin{cases} 1 - \sqrt{2(1 - v)} &amp; \text{if } v \geq 0.5 \ -1 + \sqrt{2v} &amp; \text{if } v &lt; 0.5 \end{cases}$</td>
</tr>
<tr>
<td>Triangle with vertices $a_0, a_1, a_2$</td>
<td>$p(a) = \frac{1}{\text{area}}$</td>
<td>$s \in [0, 1]$</td>
<td>$s = 1 - \sqrt{1 - u}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$t \in [0, 1 - s]$</td>
<td>$t = (1 - s)v$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$a = a_0 + s(a_1 - a_0) + t(a_2 - a_0)$</td>
<td></td>
</tr>
<tr>
<td>Surface of unit sphere</td>
<td>$p(\theta, \phi) = \frac{1}{4\pi}$</td>
<td>$\theta \in [0, \pi]$</td>
<td>$\theta = \arccos(1 - 2u)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\phi \in [0, 2\pi]$</td>
<td>$\phi = 2\pi v$</td>
</tr>
<tr>
<td>Sector on surface of unit sphere</td>
<td>$p(\theta, \phi)$</td>
<td>$\theta \in [\theta_1, \theta_2]$</td>
<td>$\theta = \arccos[\cos \theta_1 + u(\cos \theta_2 - \cos \theta_1)]$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\phi \in [\phi_1, \phi_2]$</td>
<td>$\phi = \phi_1 + v(\phi_2 - \phi_1)$</td>
</tr>
<tr>
<td>Interior of radius $R$ sphere</td>
<td>$p = \frac{3}{4\pi R^3}$</td>
<td>$\theta \in [0, \pi]$</td>
<td>$\theta = \arccos(1 - 2u)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\phi \in [0, 2\pi]$</td>
<td>$\phi = 2\pi v$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$R \in [0, R]$</td>
<td>$r = w^{1/3}R$</td>
</tr>
</tbody>
</table>

*The symbols $u$, $v$, and $w$ represent instances of uniformly distributed random variables ranging over $[0, 1]$.*

If the density is a one-dimensional $f(x)$ defined over the interval $x \in [a, b]$, then we can generate random numbers $\alpha_i$ that have density $f$ from a set of uniform random numbers $\xi_i$, where $\xi_i \in [0, 1]$. To do this we need the probability distribution function $F(x)$:

$$F(x) = \int_a^x f(x') \, d\mu(x'). \quad (1)$$
To get \( \alpha_i \) we simply transform \( \xi_i \):
\[
\alpha_i = F^{-1}(\xi_i),
\]  
(2)

where \( F^{-1} \) is the inverse of \( F \). If \( F \) is not analytically invertible, then numerical methods will suffice because an inverse exists for all valid probability distribution functions.

If we have a two-dimensional density \((x, y)\) defined on \([a, b; c, d]\), then we need the two-dimensional distribution function:
\[
F(x, y) = \int_c^y \int_a^x f(x', y') \, d\mu(x', y').
\]  
(3)

We first choose an \( x_i \) using the marginal distribution \( F(x, d) \), and then choose \( y_i \) according to \( F(x_i, y)/F(x_i, d) \). If \( F(x, y) \) is separable (expressible as \( g(x)h(y) \)), then the one-dimensional techniques can be used on each dimension.

As an example, to choose reflected ray directions for zonal calculations or distributed ray tracing, we can think of the problem as choosing points on the unit sphere or hemisphere (since each ray direction can be expressed as a point on the sphere). For example, suppose that we want to choose rays according to the density
\[
p(\theta, \phi) = \frac{n + 1}{2\pi} \cos^n \theta,
\]  
(4)

where \( n \) is a Phong-like exponent; \( \theta \) is the angle from the surface normal and \( \theta \in [0, \pi/2] \) (is on the upper hemisphere); and \( \phi \) is the azimuthal angle \( (\phi \in [0, 2\pi]) \). The distribution function is
\[
P(\theta, \phi) = \int_0^\phi \int_0^\theta p(\theta', \phi') \sin \theta' \, d\theta' \, d\phi'.
\]  
(5)

The \( \sin \theta' \) term arises because \( d\omega = \sin \theta \, d\theta \, d\phi \) on the sphere. When the marginal densities are found, \( p \) (as expected) is separable, and we find that a \((r_1, r_2)\) pair of uniform random numbers can be transformed to a direction by
\[
(\theta, \phi) = \left(\arccos((1 - r_1)^{1/(n+1)}), 2\pi r_2\right).
\]  
(6)
To get $\alpha_i$ we simply transform $\xi_i$:

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