Subsequent, we can write:

Proof.

Lemma 1. Let $Ω ∈ \mathbb{C}^{n×n}$ be a skew-hermitian matrix, i.e. $Ω^∗ = −Ω$ where $Ω^∗$ denotes the conjugate transpose of $Ω$. Then all eigenvalues of $Ω$ are imaginary (or zero).

Proof. Let $(q, λ)$ be an eigenvector-eigenvalue pair. Then

$$Ωq = λq ⇒ q^∗Ωq = λq^∗q = λ||q||^2$$

Taking the conjugate transpose of the equation above, we have

$$(q^∗Ωq)^* = (λ||q||^2)^* ⇒ q^∗Ω^∗q = λ^*||q||^2 ⇒ −q^∗Ωq = λ^*||q||^2$$

Adding the two equations $λ||q||^2 = q^∗Ωq$ and $λ^*||q||^2 = −q^∗Ωq$, we have $(λ + λ^*)||q||^2 = 0$.

Since the eigenvector $q$ cannot be zero, we have $λ + λ^* = 0$, thus $λ$ is an imaginary number (or zero).

Lemma 2. If $Ω ∈ \mathbb{R}^{n×n}$ is a skew-symmetric matrix, we can write $Ω = Λ = UΛ^∗U^T = UΛU^∗$ where $U ∈ \mathbb{C}^{n×n}$ is a unitary matrix and $Λ$ is a diagonal matrix containing entries that

(i) are all imaginary (or zero), and

(ii) come in conjugate pairs $−αi, +αi, −βi, +βi, −γi, +γi…$ where $α, β, γ… ∈ \mathbb{R}.$

Proof. Because $Ω$ is skew-symmetric, $Ω^TΩ = (−Ω)(−Ω^T) = ΩΩ^T$, i.e. $Ω$ is normal. By the spectral theorem $Ω$ is diagonalizable by a unitary matrix $U$ (with $U^T = U^{-1}$), i.e.

$$Ω = UΛU^∗ = UΛU$$

$Ω$ and $Λ$ are similar, thus the diagonal entries of $Λ$ are the eigenvalues of $Ω$. By Lemma 1, they are all imaginary (or zero).

Since $Ω ∈ \mathbb{R}^{n×n}$, all eigenvalues come in complex conjugate pairs.

Lemma 3. If $S ∈ \mathbb{R}^{n×n}$ is a symmetric positive definite matrix, and $Λ ∈ \mathbb{R}^{n×n}$ is a skew-symmetric matrix, then $det(S + Λ) ≥ det(S)$.

Proof. Since $S$ is symmetric positive definite, it can be written in the form $S = NN^T$ where $N ∈ \mathbb{R}^{n×n}$ (e.g. from Cholesky factorization). Subsequently, we can write:

$$det(S + Λ) = det(NN^T + Λ) = det(N(I + N^{-1}AN^{-T})N^T) = det(N)det(I + N^{-1}AN^{-T})det(N^T) = det(NN^T)det(I + N^{-1}AN^{-T}) = det(S)det(I + Ω)$$

where $Ω := N^{-1}AN^{-T}$. $Ω$ is in fact skew symmetric:

$$Ω^T = (N^{-T})^T Λ^T (N^{-1})^T = −N^{-1}AN^{-T} = −Ω$$
Thus by Lemma 2 we can write

\[
\det(I + \Omega) = \det(UU^{-1} + U\Lambda U^{-1}) \\
= \det(U) \det(I + \Lambda) \det(U^{-1}) \\
= \det(I + \Lambda)
\]

\(I + \Lambda\) is diagonal with paired imaginary entries \(-\alpha_i, +\alpha_i, -\beta_i, +\beta_i, -\gamma_i, +\gamma_i, \ldots (\alpha, \beta, \gamma \ldots \in \mathbb{R})\). Taking the product of those yields a greater or equal than 1 result since \((1 + \alpha_i)(1 - \alpha_i) = 1 + \alpha^2 \geq 1\), etc. Hence \(\det(I + \Omega) \geq 1\). This result, combined with equation 1 yields \(\det(S + A) \geq \det(S)\).

\[\square\]

**Theorem 4.** Let \(\hat{R} \in \mathbb{R}^{n \times n}\) be a rotation matrix, i.e. \(\hat{R}\) is orthonormal and \(\det(\hat{R}) = 1\), and let \(F \in \mathbb{R}^{n \times n}\).

Define \(S = \text{sym}(\hat{R}^T F)\). If \(S \succ 0\), then \(\det(F) \geq \det(S) > 0\).

**Proof.** The inequality \(\det(S) > 0\) is trivial if \(S\) is positive definite. Since \(\hat{R}\) is a rotation matrix, we have \(\det(F) = \det(\hat{R}) \det(F) = \det(\hat{R}^T F)\). Thus if we define \(M = \hat{R}^T F\), the theorem becomes equivalent to proving \(\det(M) \geq \det(S)\).

Write \(M = S + A\), where \(S = (M + M^T)/2\) the symmetric part of \(M\) as previously defined, while \(A = (M - M^T)/2\) is the skew-symmetric part of the same matrix. If \(S \succ 0\), then by Lemma 3 we have \(\det(M) = \det(S + A) \geq \det(S)\) which completes our proof. \[\square\]

## 2 Proof of convexity for our penalty energy term

Finally, we provide a proof for the convexity of the penalty term \(E_{\text{penalty}}(x) = \sum_{i,j} p(\lambda_j(S_{i,j}))\) used in our method.

**Lemma 5.** For \(p: \mathbb{R}^1 \rightarrow \mathbb{R}^1\) being a \(C^1\) continuous and convex function, for \(\forall x_1, x_2 \in \mathbb{R}^1\),

\[
(p'(x_1) - p'(x_2))(x_1 - x_2) \geq 0
\]

**Proof.** The follows directly from the fact that the derivative \(p'(x)\) is monotonically non-decreasing (due to the convexity of \(p\)). \[\square\]

**Lemma 6.** For any square matrices \(A, B\), and orthogonal matrix \(Q\):

\[
A : B = (Q^T A Q) : (Q^T B Q)
\]

where \(A : B = \sum_{i,j} a_{ij} b_{ij}\)

**Proof.** Because \(Q\) is orthogonal, \(QQ^T = Q^T Q = I\). Thus

\[
A : B = \text{tr}(AB^T) \\
= \text{tr}(AQQ^T B^T QQ^T) \\
= \text{tr}(Q^T QA \cdot Q^T B^T Q) \quad \text{(cyclic permutation invariance of trace)} \\
= (Q^T QA) : (Q^T B Q)
\]

\[\square\]

**Lemma 7.** For any square matrices \(A, B\), if \(A\) is a diagonal matrix,

\[
A : B = A : \text{diag}(B)
\]

**Proof.** \(A : B = \sum_{i,j} a_{ij} b_{ij} + \sum_{i \neq j} a_{ij} b_{ij}\). Because \(a_{ij} = 0\) for \(i \neq j\), we have

\[
A : B = \sum_{i,j} a_{ij} b_{ij} = A : \text{diag}(B)
\]

\[\square\]
Theorem 8. \( E_{\text{penalty}}(x) = \sum_{i,j} p(\lambda_j(S_i)) \) is a convex function when \( p \) is a \( C^1 \) continuous and convex function, where:

1. \( i = 1, 2, 3 \ldots m \) and \( j = 1, 2 \ldots d \).
2. \( m \) is the number of elements in the mesh, \( d \) is the dimension (\( d = 2 \) for 2D or \( d = 3 \) for 3D) of the problem.
3. \( S_i = \text{sym}(\tilde{R}_i^T F_i) \), \( \tilde{R}_i \) and \( F_i \) are the re-rotation field and deformation gradient of the \( i \)-th element respectively.
4. \( \lambda_j(S_i) \) maps from matrix \( S_i \) to its corresponding eigenvalues \( \{\lambda_1, \lambda_2, \ldots, \lambda_d\} \).

Proof. An sufficient condition to prove \( E_{\text{penalty}}(x) \) being a convex function is that \( E_{\text{penalty},i}(x) = \sum_j p(\lambda_j(S_i)) \) being a convex function for \( \forall i \). To make the notation simpler, we will discard the subscript \( i \), and write \( S = \text{sym}(\tilde{R}^T F) \), \( \Lambda = \begin{bmatrix} \lambda_1(S) & & \\ & \lambda_2(S) & \\ & & \ddots \end{bmatrix} \).

Notice that now we want to prove \( E_{\text{penalty},i} = \varphi(\Lambda(S(x))) = \sum_j p(\lambda_j(S)) \) is a convex function over \( x \). Because \( S \) is a linear mapping of \( x \), it is sufficient to just prove \( \varphi(\Lambda(S)) \) is convex over \( S \), so problem turns to be:

\[
\delta S : \frac{\partial^2 \varphi(\Lambda(S))}{\partial S^2} : \delta S \geq 0
\]

or

\[
\delta S \left( \frac{\partial \varphi(\Lambda(S))}{\partial S} \right) : \delta S \geq 0
\]

Let’s take a look at \( \frac{\partial \varphi(\Lambda(S))}{\partial S} \) first:

\[
\delta S \varphi(\Lambda) = \nabla \varphi(\Lambda) : \delta S (\Lambda)
\]

\[
\nabla \varphi(\Lambda) = \begin{bmatrix} p'(\lambda_1) & & \\ & p'(\lambda_2) & \\ & & \ddots \end{bmatrix}
\]

Since \( \Lambda \) comes from an eigen decomposition from \( S \), \( QAQ^T = S \), we have

\[
\delta S QAQ^T + Q \delta S AQ^T + QA \delta S Q^T = \delta S
\]

\[
Q^T(\delta S QAQ^T + Q \delta S AQ^T + QA \delta S Q^T)Q = Q^T \delta S Q
\]

\[
(Q^T \delta S Q)\Lambda + \delta S \Lambda + (Q^T \delta S Q)^T = Q^T \delta S Q
\]

Notice that \( QQ^T = I \),

\[
(Q^T \delta S Q)^T + Q^T \delta S Q = 0
\]

Thus, \( Q^T \delta S Q \) is a skew-symmetric matrix, and \( (Q^T \delta S Q)\Lambda + \Lambda (Q^T \delta S Q)^T \) would be an off-diagonal matrix. Hence \( \delta S \Lambda = \text{diag}(Q^T \delta S Q) \).

Therefore,

\[
\delta S(\varphi(\Lambda(S))) = \nabla \varphi(\Lambda) : \delta S \Lambda
\]

\[
= \nabla \varphi(\Lambda) : \text{diag}(Q^T \delta S Q)
\]

\[
= \nabla \varphi(\Lambda) : Q^T \delta S Q
\]

\[
= Q \nabla \varphi(\Lambda) Q^T : \delta S \quad \text{(Lemma 7)}
\]

That’s to say : \( \frac{\partial \varphi(\Lambda(S))}{\partial S} = Q \nabla \varphi(\Lambda) Q^T \) by definition. Now let’s prove \( \delta S \left( \frac{\partial \varphi(\Lambda(S))}{\partial S} \right) : \delta S \geq 0 : \)
\[
\delta_S \left( \frac{\partial \varphi(\mathbf{A})}{\partial \mathbf{S}} \right) : \delta \mathbf{S} = \delta_S(\nabla \varphi(\mathbf{A}) \mathbf{Q}^T) : \delta \mathbf{S} \\
= \delta_S(\mathbf{Q} \nabla \varphi(\mathbf{A}) \mathbf{Q}^T) : \delta_S(\mathbf{Q} \mathbf{A} \mathbf{Q}^T) \\
= (\mathbf{Q}^T \delta_S(\nabla \varphi(\mathbf{A}) \mathbf{Q}^T) \mathbf{Q}) : (\mathbf{Q}^T \delta_S(\mathbf{Q} \mathbf{A} \mathbf{Q}^T) \mathbf{Q}) \\
= ((\mathbf{Q}^T \delta_S \mathbf{Q}) \nabla \varphi(\mathbf{A}) + \delta_S(\nabla \varphi(\mathbf{A})) + \nabla \varphi(\mathbf{A}) (\mathbf{Q}^T \delta_S \mathbf{Q})^T) \\
: ((\mathbf{Q}^T \delta_S \mathbf{Q}) \mathbf{A} + \delta_S \mathbf{A} + \mathbf{A} (\mathbf{Q}^T \delta_S \mathbf{Q})^T) \\
\text{(Lemma 6)}
\]

Notice that \( \mathbf{Q}^T \delta_S \mathbf{Q} \) is a skew-symmetric matrix, we can group the diagonal terms and off-diagonal terms separately, thus

\[
\delta_S \left( \frac{\partial \varphi(\mathbf{A})}{\partial \mathbf{S}} \right) : \delta \mathbf{S} = ((\mathbf{Q}^T \delta_S \mathbf{Q}) \nabla \varphi(\mathbf{A}) + \nabla \varphi(\mathbf{A}) (\mathbf{Q}^T \delta_S \mathbf{Q})^T) : ((\mathbf{Q}^T \delta_S \mathbf{Q}) \mathbf{A} + \delta_S \mathbf{A} + \mathbf{A} (\mathbf{Q}^T \delta_S \mathbf{Q})^T) \\
\text{(\ast)}
\]

If we write down the skew-symmetric matrix \( \mathbf{Q}^T \delta_S \mathbf{Q} \) explicitly as

\[
\mathbf{Q}^T \delta_S \mathbf{Q} = \begin{bmatrix}
0 & q_{12} & \cdots & q_{1d} \\
-q_{12} & 0 & \cdots & \cdot \\
\cdot & \cdot & \ddots & \cdot \\
-q_{1d} & \cdots & -q_{d-1,d} & 0
\end{bmatrix},
\]

we can expand (\ast) to

\[
(\ast) = \begin{bmatrix}
0 & (p'(\lambda_2) - p'(\lambda_1))q_{12} & \cdots & (p'(\lambda_d) - p'(\lambda_1))q_{1d} \\
(p'(\lambda_2) - p'(\lambda_1))q_{12} & 0 & \cdots & \cdot \\
\cdot & \cdot & \ddots & \cdot \\
(p'(\lambda_d) - p'(\lambda_1))q_{1d} & \cdots & (p'(\lambda_d) - p'(\lambda_{d-1}))q_{d-1,d} & 0
\end{bmatrix}
\]

Since function \( p \) is \( C^1 \) continuous and convex, we have \( (p'(\lambda_2) - p'(\lambda_1))(\lambda_1 - \lambda_k) \geq 0 \) by applying Lemma 5, thus \((\ast) \geq 0\).

Similarly, we can expand (\ast\ast) to

\[
(\ast\ast) = \sum_{k=1}^{d} p''(\lambda_k)(\delta_S \lambda_k)^2
\]

Once again because \( p \) is a convex function, \( p''(\lambda_k) \geq 0 \). Thus \( (\ast\ast) \geq 0 \).

Therefore, we proved that \( \delta_S \left( \frac{\partial \varphi(\mathbf{A})}{\partial \mathbf{S}} \right) : \delta \mathbf{S} \geq 0 \), and \( E_{\text{penalty}}(\mathbf{x}) = \sum_{i,j} p(\lambda_j(S_i)) \) is a convex function. 
\( \square \)