L10: Spectral Clustering

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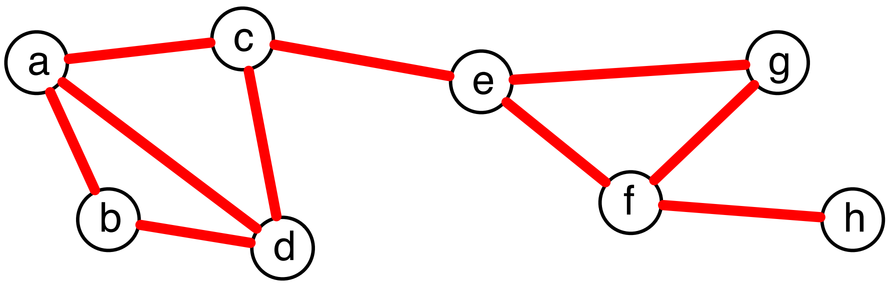
Graphs

c ∈ (0.7, 0.5)

$|E| = |V|^{1+c}$



Mathematically: $G = (V, E)$ where

$V = \{a, b, c, d, e, f, g\}$ and

$$E = \Big\{\{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{c, d\}, \{c, e\}, \{e, f\}, \{e, g\}, \{f, g\}, \{f, h\}\Big\}.$$

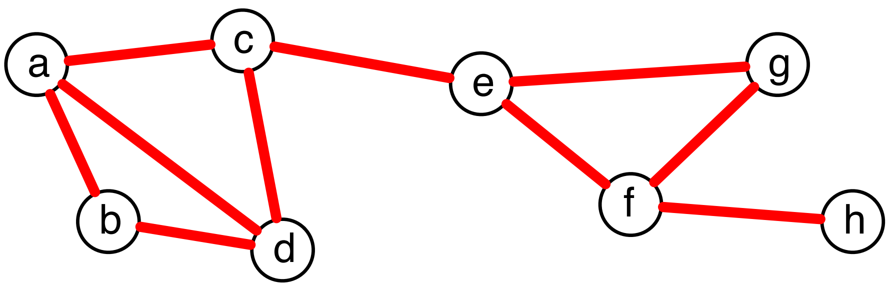
Matrix-Style: As a matrix with 1 if there is an edge, and 0 otherwise. (For a directed graph, it may not be symmetric).

$G =$

	a	*b*	*c*	*d*	*e*	*f*	*g*	*h*
a	0	1	1	1	0	0	0	0
b	1	0	0	1	0	0	0	0
c	1	0	0	1	1	0	0	0
d	1	1	1	0	0	0	0	0
e	0	0	1	0	0	1	1	0
f	0	0	0	0	1	0	1	1
g	0	0	0	0	1	1	0	0
h	0	0	0	0	0	1	0	0

$$= \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Laplacian Matrix



Sim(a,b) → [0,1]

Affinity Matrix

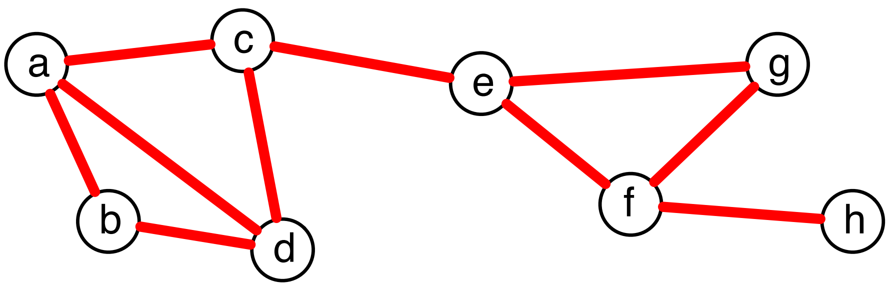
Kernel

$K(a,b) = e^{-\frac{\|a-b\|^2}{\sigma^2}}$

adjacency (Gram) diagonal (degree)

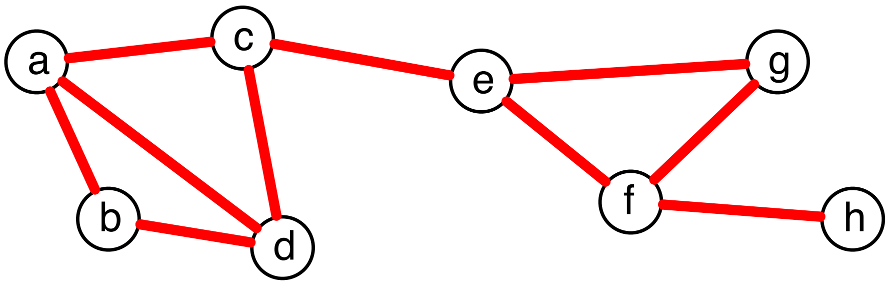
$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Unnormalized Laplacian Matrix



$$L_0 = D - A = \begin{pmatrix} 3 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 3 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 3 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}.$$

Unnormalized Laplacian Matrix



$Mv = \lambda v$

↑ matrix ↑ scalar

eigenvectors of L_0

λ_2

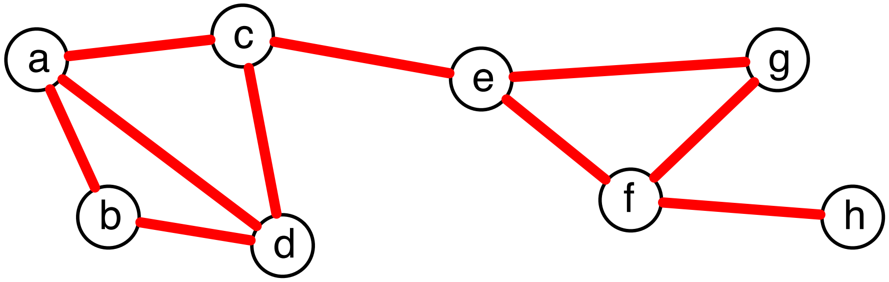
λ	0	**0.278**	1.11	2.31	3.46	4	4.82
V	$1/\sqrt{8}$	$-.36$	0.08	0.10	0.28	0.25	$1/\sqrt{2}$
	$1/\sqrt{8}$	$-.42$	0.18	0.64	$-.38$	0.25	0
	$1/\sqrt{8}$	$-.20$	$-.11$	0.61	0.03	$-.25$	0
	$1/\sqrt{8}$	$-.36$	0.08	0.10	0.28	0.25	$-1/\sqrt{2}$
	$1/\sqrt{8}$	0.17	$-.37$	0.21	$-.54$	$-.25$	0
	$1/\sqrt{8}$	0.36	$-.08$	$-.10$	$-.28$	0.75	0
	$1/\sqrt{8}$	0.31	$-.51$	$-.36$	$-.56$	0.56	0
	$1/\sqrt{8}$	0.50	0.73	0.08	0.11	0.11	0

$(V, L) = eig(L)$

v_2

Feidler vector

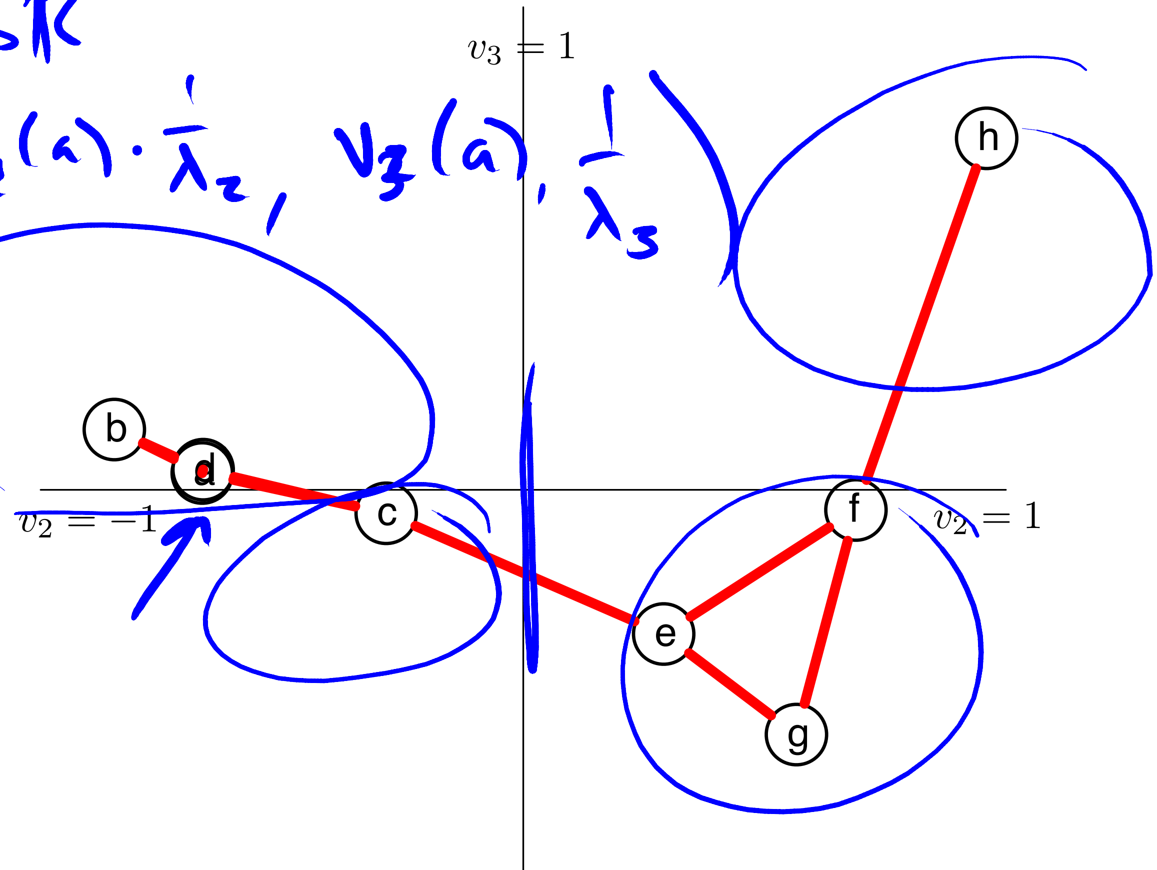
Unnormalized Laplacian Matrix

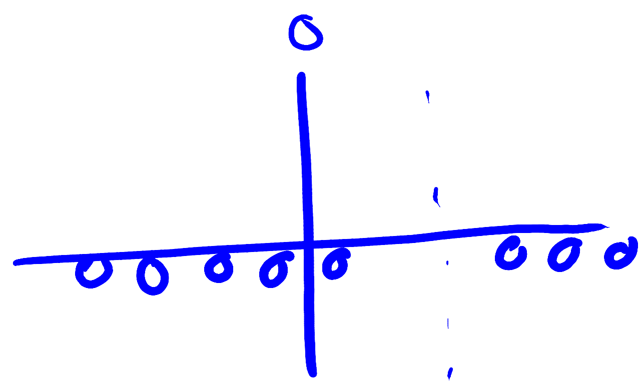


λ	**0.278**	1.11	
V	−.36	0.08	*a*
	−.42	0.18	*b*
	−.20	−.11	*c*
	−.36	0.08	*d*
	0.17	−.37	*e*
	0.36	−.08	*f*
	0.31	−.51	*g*
	0.50	0.73	*h*
	v_2	v_3	

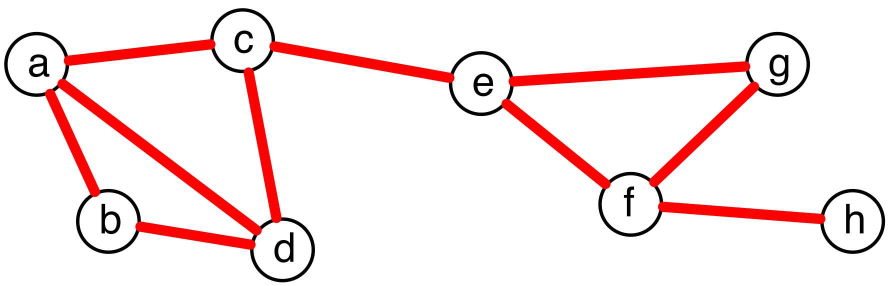
$a \to \mathbb{R}^2$

$\left(v_2(a)\cdot\frac{1}{\lambda_2},\ v_3(a)\cdot\frac{1}{\lambda_3}\right)$



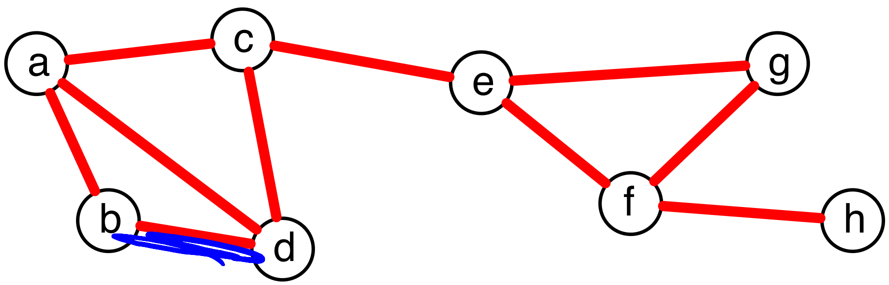


Laplacian Matrix



$$D^{1/2} = \begin{pmatrix} 0.577 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.707 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.577 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.577 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.577 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.577 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.707 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Laplacian Matrix



$D - A$

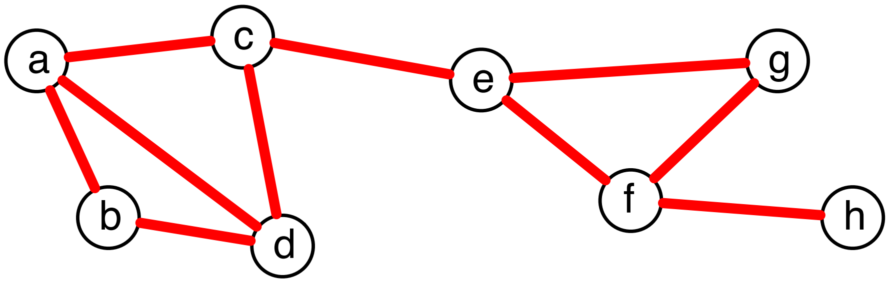
normalized Laplacian

$$L = I - D^{1/2} A D^{1/2} = D^{-1/2}(L_0)D^{-1/2}$$

$$\begin{pmatrix} 1 & -0.408 & -0.333 & -0.333 & 0 & 0 & 0 & 0 \\ -0.408 & 1 & 0 & -0.408 & 0 & 0 & 0 & 0 \\ -0.333 & 0 & 1 & -0.333 & -0.333 & 0 & 0 & 0 \\ -0.333 & -0.408 & -0.333 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.333 & 0 & 1 & -0.333 & -0.408 & 0 \\ 0 & 0 & 0 & 0 & -0.333 & 1 & -0.408 & -0.577 \\ 0 & 0 & 0 & 0 & -0.408 & -0.408 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.577 & 0 & 1 \end{pmatrix}.$$

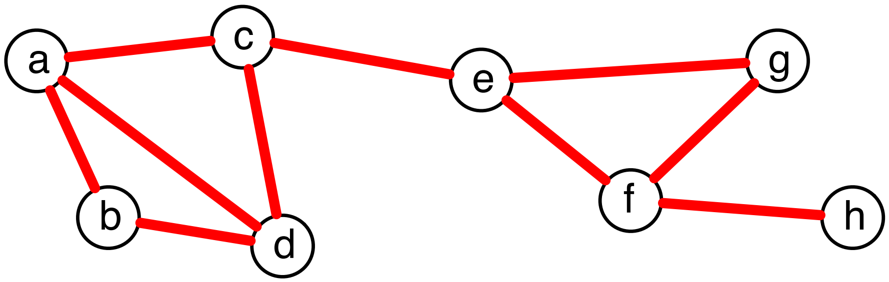
$$\frac{-1}{\sqrt{d_b \cdot d_d}}$$

Laplacian Matrix



eigenvectors of L

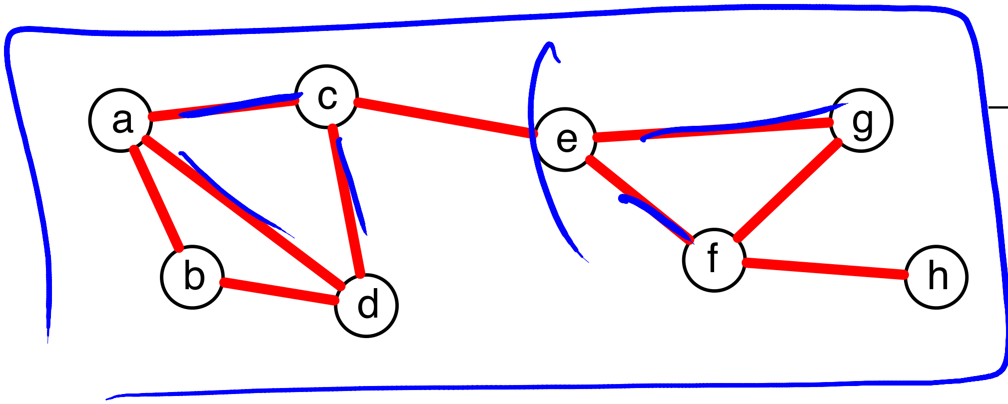
Laplacian Matrix



eigenvectors of L

λ	0	**0.125**	0.724	1.00	1.33	1.42	1.66	1.73
V	−.39	0.38	−.09	0.00	0.71	0.26	−.32	0.16
	−.32	0.36	−.27	0.50	0.00	−.51	0.38	−.18
	−.39	0.18	0.36	−.61	0.00	0.03	0.47	−.29
	−.39	0.38	−.09	0.00	−.71	0.26	−.32	0.16
	−.39	−.28	0.48	0.00	0.00	−.57	0.31	0.33
	−.39	−.48	−.29	0.00	0.00	0.05	−.31	−.65
	−.31	−.36	0.27	0.50	0.00	0.51	0.38	−.18
	−.22	−.32	−.61	−.35	0.00	−.07	0.27	0.51

Laplacian Matrix



λ	**0.125**	0.724	
V	0.38	$-.09$	*a*
	0.36	$-.27$	*b*
	0.18	0.36	*c*
	0.38	$-.09$	*d*
	$-.28$	0.48	*e*
	$-.48$	$-.29$	*f*
	$-.36$	0.27	*g*
	$-.32$	$-.61$	*h*
	v_2	v_3