Homework 2: Convergence and Linear Algebra

Instructions: Your answers are due at 11:50pm submitted on canvas. You must turn in a pdf through canvas. I recommend using latex (http://www.cs.utah.edu/~jeffp/teaching/latex/, see also http://overleaf.com) for producing the assignment answers. If the answers are too hard to read you will lose points, entire questions may be given a 0 (e.g. sloppy pictures with your phone’s camera are not ok, but very careful ones are)

Please make sure your name appears at the top of the page.

You may discuss the concepts with your classmates, but write up the answers entirely on your own. Be sure to show all the work involved in deriving your answers! If you just give a final answer without explanation, you may not receive credit for that question.

1. [30 points] Consider a random variable $X$ with expected values $E[X] = 10$ and variance $\text{Var}[X] = 2$. We would like to upper bound the probability $\Pr[X > 12]$.
   
   (a) Which bound can and cannot be used with what we know about $X$ (Markov, Chebyshev, or Chernoff-Hoeffding), and why?
   
   (b) Using that bound, calculate an upper bound for $\Pr[X > 12]$.
   
   (c) Describe a probability distribution for $X$ where the other two bounds are definitely not applicable.

2. [30 points] Consider a pdf $f$ so that a random variable $X \sim f$ has expected value $E(X) = 2$ and variance $\text{Var}(X) = 4$. Now consider $n = 10$ iid random variables $X_1, X_2, \ldots, X_{10}$ drawn from $f$. Let $\bar{X} = \frac{1}{10} \sum_{i=1}^{10} X_i$

   (a) What is $E(\bar{X})$?
   
   (b) What is $\text{Var}(\bar{X})$?

   Assume we know $X$ is never smaller than 0 and never larger than 7

   (c) Use Markov inequality to upper bound $\Pr(\bar{X} > 3)$
   
   (d) Use Chebyshev inequality to upper bound $\Pr(\bar{X} > 3)$
   
   (e) Use Chernoff-Hoeffding inequality to upper bound $\Pr(\bar{X} > 3)$
   
   (f) Now suppose $n = 100$. Calculate the 3 bounds again. For this part, also make sure to report the name of the inequality that gives the tightest bound, the second tightest bound, and the third tightest bound respectively.

3. [15 points] Consider the following 2 vectors in $\mathbb{R}^4$:

   $p = (2, -4, 8, x)$
   
   $q = (4, -8, 16, -10)$

   Report the following:
(a) Choose the value $x$ so that $p$ and $q$ are linearly dependent
(b) Choose the value $x$ so that $p$ and $q$ are orthogonal
(c) Choose a single value of $x$ so that $p$ and $q$ are neither linearly dependent nor orthogonal
(d) Calculate $\|q\|_1$
(e) Calculate $\|q\|_2^2$

4. **[25 points]** Consider the following 2 matrices:

$$
A = \begin{bmatrix}
2 & -1 & 4 \\
0 & -1 & 0 \\
3 & -2 & 6
\end{bmatrix}
\quad B = \begin{bmatrix}
0 & 1 & 0 \\
2 & 0 & 0 \\
0 & 0 & 3
\end{bmatrix}
$$

Report the following (e.g using Python):

(a) $A^T B$
(b) $AB$
(c) $BA$
(d) $B + A$
(e) $B^T$
(f) Which matrices are invertable? For any that are invertable, report the result.