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An improved, randomized algorithm for parallel selection with an experimental study[☆]

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Abstract

A common statistical problem is that of finding the median element in a set of data. This paper presents an efficient randomized high-level parallel algorithm for finding the median given a set of elements distributed across a parallel machine. In fact, our algorithm solves the general selection problem that requires the determination of the element of rank k, for an arbitrarily given integer k.

Our general framework is an SPMD distributed-memory programming model that is enhanced by a set of communication primitives. We use efficient techniques for distributing and coalescing data as well as efficient combinations of task and data parallelism. The algorithms have been coded in the message-passing standard MPI, and our experimental results from the IBM SP-2 illustrate the scalability and efficiency of our algorithm and improve upon all the related experimental results known to the author.

The main contributions of this paper are

(1) New techniques for speeding the performance of certain randomized algorithms, such as selection, which are efficient with likely probability.

(2) A new, practical randomized selection algorithm (UltraFast) with significantly improved convergence.

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1. Introduction

Selection and median finding in large data sets are important statistical measures needed by a variety of high-performance computing applications, for example, image processing for computer vision and remote sensing, computational aerodynamics and physics simulations, and data mining of large databases. In these applications, the data set typically is already evenly distributed across the processing nodes. Because of the large data volume, solving the problem sequentially surely would overwhelm a single processor.

Given a set of data X with $|X| = n$, the selection problem requires the determination of the element with rank k (that is, the kth smallest element), for an arbitrarily given integer k. Median finding is a special case of selection with $k = \frac{n}{2}$. In previous work, we have designed deterministic and efficient parallel algorithms for the selection problem on current parallel machines [3,6,7]. In this paper, we discuss a new UltraFast Randomized algorithm for the selection problem that, unlike previous research (for example, [10,12,14–21]), is not dependent on network topology or limited to the PRAM model which does not assign a realistic cost for communication. In addition, our randomized algorithm improves upon previous implementations on current parallel platforms, for example, [2] implements both our deterministic algorithm and the

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randomized algorithms due to Rajasekaran et al. (e.g., [15,17]) on the TMC CM-5. An earlier version of this paper appeared as [5].

The main contributions of this paper are:

(1) New techniques for speeding the performance of certain randomized algorithms, such as selection, that are efficient with likely probability.
(2) A new, practical randomized selection algorithm (UltraFast) with significantly improved convergence.

The remainder of this paper is organized as follows. Our new randomized selection algorithm is detailed in Section 2, followed by analysis and experimental results in Section 3. More extensive statistics from our experiments are reported in [4].

2. Parallel selection

The selection algorithm for rank k assumes that input data X of size n are initially distributed evenly across the p processors, such that each processor holds $\frac{n}{p}$ elements. The output, namely the element from X with rank k, is returned on each processor.

The randomized selection algorithm locates the element of rank k by pruning the set of candidate elements using the following iterative procedure. Two *splitter* elements (k_1, k_2) are chosen that partition the input into three groups, G_0, G_1, and G_2, such that each element in G_0 is less than k_1, each element in G_1 lies in $[k_1, k_2]$, and each in G_2 is greater than k_2. The desire is to have the middle group G_1 much smaller than the outer two groups $(|G_1| \ll |G_0|, |G_2|)$ with the *condition* that the selection index lies within this middle group. The process is repeated iteratively on the group holding the selection index until the size of the group is "small enough," whereby the remaining elements are gathered onto a single processor and the problem is solved sequentially.

The key to this approach is choosing splitters k_1 and k_2 that minimize the size of the middle group while maximizing the probability of the *condition* that the selection index lies within this group. Splitters are chosen from a random sample of the input, by finding a pair of elements of certain rank in the sample (see Section 3). The "Fast Randomized" algorithm of Rajasekaran and Reif (see [15,17] and implemented in [2]) takes a conservative approach that guarantees the condition with high probability. We have discovered a more aggressive technique for pruning the input space by choosing splitters closer together in the sample while holding the condition with likely probability. In practice, the condition almost always holds, and in the event of a failure, new splitters are chosen from the sample with a greater spread of ranks until the condition is satisfied.

In addition, we improve upon previous algorithms in the following ways.

(1) *Stopping criterion*. For utmost performance, current parallel machines require a coarse granularity, the measure of problem size per node, because communication is typically an order of magnitude slower than local computation. In addition, machine configurations tend to be small to moderate in terms of number of processors (p). Thus, a stopping criterion of problem size $< p^2$ is much too fine grained for current machines, and we suggest, for instance, a stopping size of $\max(p^2, 4096)$. When p is small and $n = O(p^2)$, a second practical reason for increasing the stopping size is that the sample is very limited and might not yield splitters that further partition the input.
(2) *Aggressive convergence*. As outlined in Section 3, our algorithm converges roughly twice as fast as the best known previous algorithm.
(3) *Algorithmic reduction*. At each iteration, we use "selection" to choose the splitters instead of sorting, a computationally harder problem.
(4) *Communication aggregation*. Similar collective communication steps are merged into a single operation. For instance, instead of calling the *Combine* primitive twice to find the size of groups G_0 and G_1 ($|G_2|$ can be calculated from this information and the problem size), we aggregate these operations into a single step.
(5) *"Min/Max" selection algorithm*. When the selection index is relatively close to 1 or n, our approach switches to a faster algorithm for this special case.

Next we outline our new UltraFast randomized selection algorithm (see Algorithm 1).

3. Analysis

The following sampling lemma from Rajasekaran (see [17]) will be used in the analysis.

Let $S = \{v_1, v_2, \ldots, v_s\}$ be a random sample from a set X of cardinality n. Also, let $v'_1, v'_2, \ldots, v'_s$ be the sorted order of this sample. If r_i is the rank of k'_i in X, the following lemma provides a high probability confidence interval for r_i.

Lemma 1. *For every* α, $Pr\left(|r_i - i\frac{n}{s}| > \sqrt{3\alpha}\frac{n}{\sqrt{s}}\sqrt{\ln n}\right) < n^{-\alpha}$.

Thus, if k_1 and k_2 are chosen as the splitters from sample set S by selecting the elements with rank $\frac{is}{n} - d\sqrt{s \ln n}$ and $\frac{is}{n} + d\sqrt{s \ln n}$, respectively, and $d = \sqrt{4\alpha}$, then the element of desired rank will lie in the middle partition $(c_{less}, c_{less} + c_{mid}]$ with high probability $(1 - n^{-\alpha})$.

A tradeoff occurs between the size of the middle partition (r) and the confidence that the desired element lies within this partition. Note that in the Fast Randomized algorithm, with $d = 1$, this probability is $1 - n^{-\frac{1}{4}}$, and $r \leqslant 8\frac{n}{\sqrt{s}}\sqrt{\ln n}$. Since $s \approx n^{\varepsilon}$, this can be approximated by $r \leqslant 8n^{1-\frac{\varepsilon}{2}}\sqrt{\ln n}$.

Algorithm 1. *UltraFast Randomized Selection Algorithm for processor P_i*

```
Input
  {$n$} - Total number of elements            {$p$} - Total number of processors, labeled from 0 to $p-1$
  {$L_i$}- List of elements on processor $P_i$, where $|L_i| = \frac{n}{p}$
  {$C$}-A constant $\approx \max(p^2, 4096)$          {$\varepsilon$}- $\log_n$ of the sample size (e.g. 0.6)
  {$\Delta^*$}-selection coefficient (e.g. 1.0)      {$\kappa$}-selection coefficient multiplier (e.g. 2.25)
  {$\eta$}- Min/Max constant (e.g. $2p$)              rank-desired rank among the elements
begin
  Step 0. Set $n_i = \frac{n}{p}$.
  While $(n > C)$ and $(|n - rank| > \eta)$
    Step 1. Collect a sample $S_i$ from $L_i$ by picking $n_i \frac{n^\varepsilon}{n}$ elements at random on $P_i$.
    Step 2. $S = \mathbf{Gather}(S_i, p)$.
    Set $z = \text{TRUE}$ and $\Delta = \Delta^*$.
    While $(z \equiv \text{TRUE})$
      On $P_0$
        Step 3. Select $k_1, k_2$ from $S$ with ranks $\left\lfloor \frac{i|S|}{n} - \Delta\sqrt{|S|} \right\rfloor$ and $\left\lfloor \frac{i|S|}{n} + \Delta\sqrt{|S|} \right\rfloor$.
        Step 4. Broadcast $k_1$ and $k_2$.
      Step 5. Partition $L_i$ into $< k_1$ and $[k_1, k_2]$, and $> k_2$, to give counts less, middle,
      (and high). Only save the elements that lie in the middle partition.
      Step 6. $c_{less} = \mathbf{Combine}(less, +)$; $c_{mid} = \mathbf{Combine}(middle, +)$;
      Step 7. If $(rank \in (c_{less}, c_{less} + c_{mid}])$
        $n = c_{mid}$; $n_i = middle$; $rank = rank - c_{less}$; $z = \text{FALSE}$
      Else
        On $P_0$: $\Delta = \kappa \cdot \Delta$
      Endif
    Endwhile
  Endwhile
  If $(|n - rank| \leqslant \eta)$ then
    If $rank \leqslant \eta$ then we use the "minimum" approach, otherwise, we use the "maximum"
    approach in parentheses, as follows.
    Step 8. Sequentially sort our $n_i$ elements in nondecreasing (nonincreasing) order using a
    modified insertion sort with output size $|L_i| = \min(rank, n_i)$ ($|L_i| = \min(n - rank + 1, n_i)$).
    An element that is greater (less) than the $L_i$ minimum (maximum) elements is discarded.
    Step 9. Gather the $p$ sorted subsequences onto $P_0$.
    Step 10. Using a $p$-way tournament tree of losers [13] constructed from the $p$ sorted
    subsequences, (rank) $(n - rank + 1)$ elements are extracted, to find the element $q$ with
    selection index rank.
  Else
    Step 11. $L = \mathbf{Gather}(L_i)$.
    Step 12. On $P_0$
      Perform sequential selection to find element $q$ of rank in $L$;
  Endif
  $result = \mathbf{Broadcast}(q)$.
end
```

Suppose now the bound is relaxed with probability no less than $1 - n^{-\alpha} = \rho$. Then $\alpha = -\frac{\log(1-\rho)}{\log n}$, and the splitters k_1, k_2 can be chosen with ranks $\frac{is}{n} - \Delta\sqrt{s}$ and $\frac{is}{n} + \Delta\sqrt{s}$, for $\Delta = 2\sqrt{-\ln(1-\rho)}$ (see Table 1). Then the size of the middle partition can be bounded similarly by $r \leqslant 16\frac{n}{\sqrt{s}}\sqrt{-\ln(1-\rho)}$. This can be approximated by $r \leqslant 16n^{1-\frac{\varepsilon}{2}}\sqrt{-\ln(1-\rho)}$. Thus, the middle partition size of the UltraFast algorithm is typically smaller than that of the Fast algorithm, whenever the condition $n > (1-\rho)^{-4}$.

A large value for ε increases running time since the sample (of size n^ε) must be either sorted (in Fast) or have elements selected from it (in UltraFast). A small value of ε increases the probability that both of the splitters lie on one side of the desired element, thus causing an unsuccessful iteration. In practice, 0.6 is an appropriate value for ε [2].

3.1. Complexity

We use a simple model of parallel computation to analyze the performance of the selection algorithms, similar to the message-passing models (e.g., BSP, LogP, etc.). Current hardware platforms can be viewed as a collection of powerful processors connected by a communication network that can be modeled as a complete graph on which communication is subject to the restrictions imposed by the latency and the bandwidth properties of the network. We view a parallel algorithm as a sequence of local computations interleaved with communication steps, and we allow computation and communication to overlap. We account for communication costs as follows.

The transfer of a block consisting of m contiguous words, assuming no congestion, takes $O(\tau + \sigma m)$ time, where τ is

Table 1
Lower bound of the capture probability (ρ) that the selection index is in the middle partition, where $\rho = 1 - e^{-\frac{\Delta^2}{4}}$

Δ	Lower bound of capture (ρ, in %)
6.07	99.99
5.26	99.9
4.29	99.0
3.03	90.0
2.54	80.0
2.19	70.0
1.91	60.0
1.50	43.0
1.00	22.1
0.50	6.05

an bound on the latency of the network and σ is the time per word at which a processor can inject or receive data from the network.

One iteration of the Fast randomized selection algorithm takes $O\left(n^{(j)} + (\tau + \sigma) \log\, p\right)$ time, where $n^{(j)}$ is the maximum number of elements held by any processor during iteration j. From the bound on the size of the middle partition, we find a recurrence on the problem size during iteration i,

$$\begin{aligned} n_0 &= n, \\ n_i + 1 &\leqslant 8 n_i^{0.7} \sqrt{\ln\, n_i}, \end{aligned} \tag{1}$$

which shows a geometric decrease in problem size per iteration, and thus, $O(\log\log n)$ iterations are required. Since $n^{(j)} = O\left(\frac{n}{p}\right)$, Fast selection requires

$$O\left(\frac{n}{p} \log\log n + (\tau + \sigma) \log\, p\, \log\log n\right) \tag{2}$$

time. Assuming random data distribution, the running time reduces to the following [2]:

$$O\left(\frac{n}{p} + (\tau + \sigma) \log\, p\, \log\log n\right). \tag{3}$$

Each iteration of the UltraFast algorithm is similar to Fast, except sorting is replaced by sequential selection, which takes linear time [11]. Also, the problem size during iteration i is bounded with the following recurrence:

$$\begin{aligned} n_0 &= n, \\ n_{i+1} &\leqslant 16 n_i^{0.7} \sqrt{-\ln(1-\rho)}, \end{aligned} \tag{4}$$

and similar to the Fast algorithm, UltraFast requires $O(\log\log n)$ iterations. Thus, UltraFast randomized selection has a similar complexity, with a worst case running time given in Eq. (2). As we will show later by empirical results in Table 3, though, the constant associated with the number of iterations is significantly smaller for the UltraFast algorithm.

3.2. Experimental data sets

Empirical results for the selection algorithm use the following five input classes. Given a problem of size n and p processors,

[I]—Identical elements $\{0, 1, \ldots, \frac{n}{p} - 1\}$ on each processor.

[S]—Sorted elements $\{0, 1, \ldots, n-1\}$ distributed in p blocks across the processors.

[R]—Random, uniformly-distributed, elements, with $\frac{n}{p}$ elements per processor.

[N]— This input is taken from the NAS Parallel Benchmark for Integer Sorting [9]. Keys are integers in the range $[0, 2^{19})$, and each key is the average of four consecutive uniformly-distributed pseudo-random numbers generated by the following recurrence:

$$x_{k+1} = a x_k (\bmod 2^{46}),$$

where $a = 5^{13}$ and the seed $x_0 = 314159265$. Thus, the distribution of the key values is a Gaussian approximation. On a p-processor machine, the first $\frac{n}{p}$ generated keys are assigned to P_0, the next $\frac{n}{p}$ to P_1, and so forth, until each processor has $\frac{n}{p}$ keys.

[K]—This input contains $\frac{n}{p}$ randomly generated elements per processor, sampled from the skewed log-normal distribution, [1] in the range of positive integers [1, INTMAX] (where INTMAX, for example, is $2^{31} - 1$ on a 32-bit machine). We generate each pseudo-random integer

$$\left(\left\lfloor \exp\left(\frac{1}{12} \ln \text{INTMAX} \cdot \text{normRand}(0,1) + \frac{1}{2} \ln \text{INTMAX}\right)\right\rfloor\right)$$

by taking the largest integer less than or equal to the exponential of a mean 0, standard deviation 1 Gaussian random number (found by adding together twelve uniformly distributed random numbers from the range $[-0.5, 0.5)$) that is first scaled by $\frac{1}{12} \ln$ INTMAX and then displaced to the right by $\frac{1}{2} \ln$ INTMAX. For a given INTMAX, the mean and standard deviation of this skewed distribution are computable. [2]

3.3. Empirical results

Results for a previous implementation of the Fast randomized selection algorithm on the TMC CM-5 parallel machine appear in [2]. However, this machine is no longer available and does not support the current message-passing standard MPI. Therefore, we have recoded this algorithm into MPI.

Table 2 compares the execution time of the Fast randomized algorithm on both the CM-5 [1,2] and the IBM SP-2.

[1] The log-normal is a distribution whose natural logarithm is a normal distribution. Given a normal distribution with mean μ and standard deviation σ, the log-normal distribution $\exp(\text{norm}(\mu, \sigma))$ has mean $e^{\mu + \sigma^2/2}$ and variance $e^{2\mu + \sigma^2}(e^{\sigma^2} - 1)$.

[2] For our generator, a log-normal distribution with mean μ and standard deviation σ, the scale ($\frac{1}{12} \ln$ INTMAX) of the mean 0, s.d. 1, Gaussian random number equals $\sqrt{\ln\left(\frac{\mu^2 + \sigma^2}{\mu^2}\right)}$, the displacement ($\frac{1}{2} \ln$ INTMAX) equals $\frac{1}{2} \ln\left(\frac{\mu^4}{\mu^2 + \sigma^2}\right)$.

Table 2
Comparison of the execution time of the Fast randomized selection algorithm on TMC CM-5 and IBM SP-2-TN (in ms)

n	p	[R]andom Input			[S]orted Input		
		CM-5 33	SP-2 66 P2	SP-2 160 P2SC	CM-5 33	SP-2 66 P2	SP-2 160 P2SC
512K	4	174	68.0	23.5	194	104	25.6
	8	105	62.7	17.2	119	79.6	21.7
	16	69.5	39.5	10.8	86.7	61.9	15.6
2M	4	591	153	56.6	601	229	67.3
	8	318	108	37.6	359	182	48.0
	16	193	74.4	23.7	237	136	34.6

Since selection is computation-bound, we would expect the performance to be closely related to the node performance of these two machines. The SP-2-TN 66 MHz Power2 (66-P2) processor is roughly twice as fast as the CM-5 33 MHz RISC processor. As expected, this factor of two performance improvement is apparent in the execution time comparison for equivalent machine and problem sizes. In actuality, the SP-2 is more than twice as powerful, since communication latency and bandwidth are improved roughly by a factor of three. The newer SP-2-TN 160 MHz Power2 SuperChip (160-P2SC) nodes are roughly three times faster than the 66-P2 nodes, and we see a similar performance improvement.

We conducted experiments with our UltraFast and the known Fast randomized selection algorithms on an IBM SP-2 (with 160-P2SC nodes) with four, eight, and sixteen processors, by finding the median of each input in the previous section for various problem sizes (ranging between $16K$ to $16M$ elements).[3] A comparison of the empirical execution times for machine configurations of $p = 4, 8,$ and 16 processors are graphed using log–log plots in Figs. 1, 2, and 3, respectively. In all cases, the UltraFast algorithm is substantially faster than the Fast randomized selection algorithm, typically by a factor of 2. Running time can be characterized mainly by $\frac{n}{p} \log p$ and is only slightly dependent on input distribution. In addition, we have included the performance of several variations as follows:

FR—the Fast Randomized algorithm from [2,15,17];

FT—the modified (and improved) Fast Randomized with the *while* loop stopping criterion of $n \leqslant \max(p^2, 4096)$ instead of $n \leqslant p^2$;

R2—the modified UltraFast Randomized algorithm without the “Min/Max” selection improvement when $(|n - rank| \leqslant \eta)$; and

R3—the UltraFast randomized algorithm (Alg. (1)).

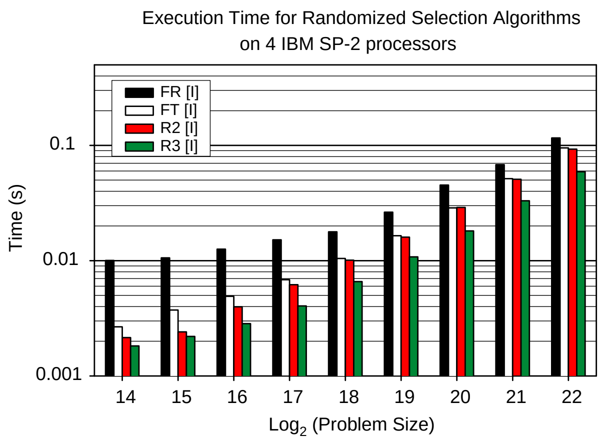
For $p = 8$, Table 3 provides a summary of the number of times each algorithm iterates. While the Fast algorithm typically iterates in the neighborhood of about 25 times, there are some cases when it iterates hundreds or even thousands of times. For some other problem instances, the Fast algorithm may encounter an infinite loop when the number of elements in a step is larger than p^2, and no choice of splitters further partitions the elements. On the other hand, the UltraFast algorithm never iterates more then three times. This is due to two reasons. First, UltraFast converges roughly twice as fast as the Fast algorithm. Second, the algorithm stops iterating by using a more realistic stopping criterion matched to the coarse granularity of current parallel machines. In addition, when p is small and $n = O(p^2)$, the Fast algorithm’s sample is very limited and sometimes does not yield splitters that further partition the input. Thus, in this situation, the Fast algorithm may iterate from tens to thousands of times before pruning any additional elements from the solution space.

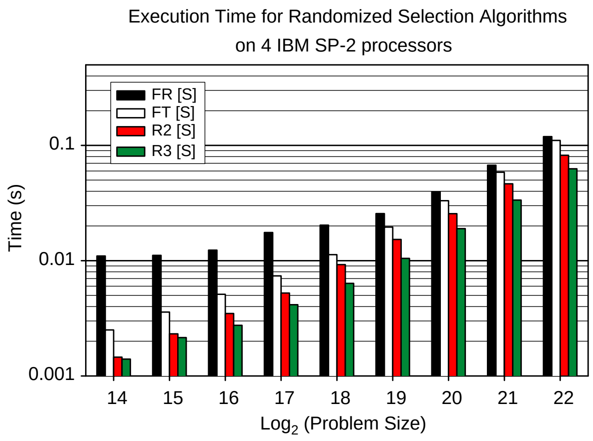
Table 3
Total number of iterations of the Fast and UltraFast randomized selection algorithms

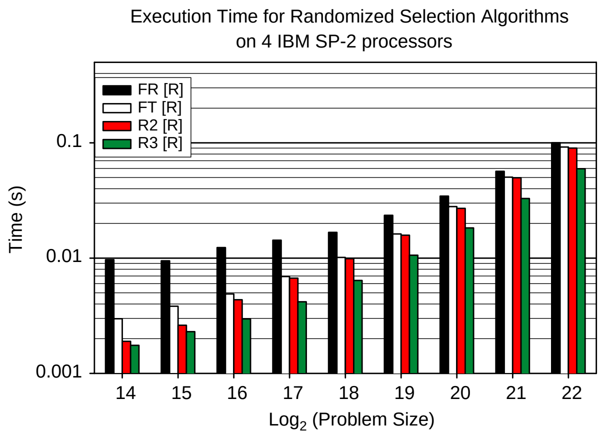
n	Input	Fast	UltraFast
512K	I	19	2
	S	17	2
	R	29	2
	N	19	2
	K	18	2
1M	I	24	2
	S	17	2
	R	22	2
	N	32	2
	K	22	2
2M	I	26	2
	S	22	3
	R	21	2
	N	38	3
	K	20	3
4M	I	37	3
	S	23	3
	R	21	3
	N	4095	3
	K	90	3
8M	I	28	3
	S	24	3
	R	21	3
	N	866	3
	K	∞	3

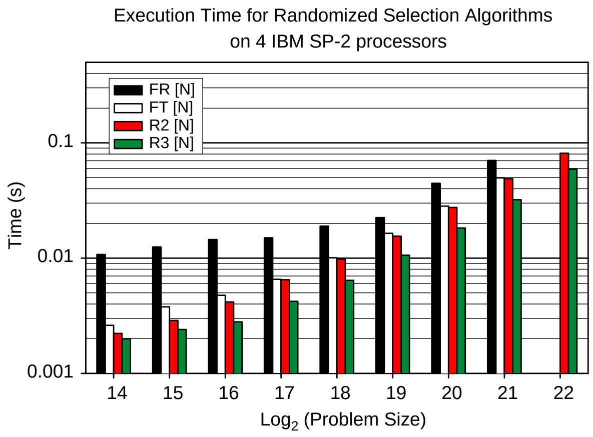
For this table, eight processors are used.

[3] Throughout this paper, K and M refer to 2^{10} and 2^{20}, respectively.









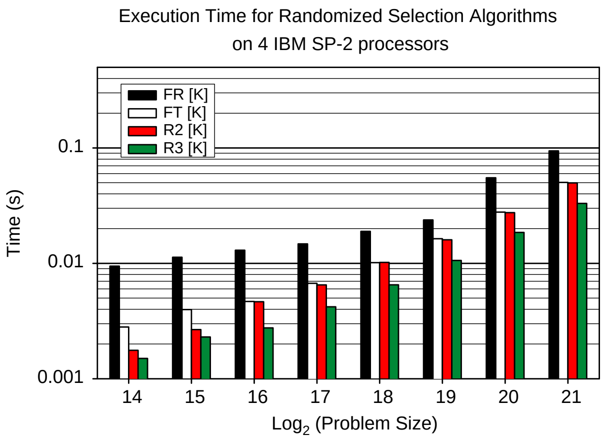


Fig. 1. Empirical performance of Fast versus UltraFast randomized selection algorithms with $p = 4$ nodes of an IBM SP-2-TN. Each of the five graphs represents one input class ([I], [S], [R], [N], and [K]) and increasing problem sizes on the x-axis. For each input class and problem size, there are four algorithms/variations reported: FR—the Fast randomized; FT—the modified Fast randomized with stopping criterion of $\max(p^2, 4096)$ instead of p^2; R2—the UltraFast randomized without the "Min/Max" selection improvement; and R3—the UltraFast randomized algorithm.

Extended tables of statistics for various inputs and machine sizes are provided for both algorithms in our technical report [4] that show for each iteration the sample size (s), partitioning information, and a term, $k^\star$, defined as half of the difference between the ranks of the two splitters selection from the sample.

4. Future directions

We are investigating other combinatorial algorithms that may have significant practical improvement by relaxing the probabilistic bounds, as demonstrated by our UltraFast randomized selection.

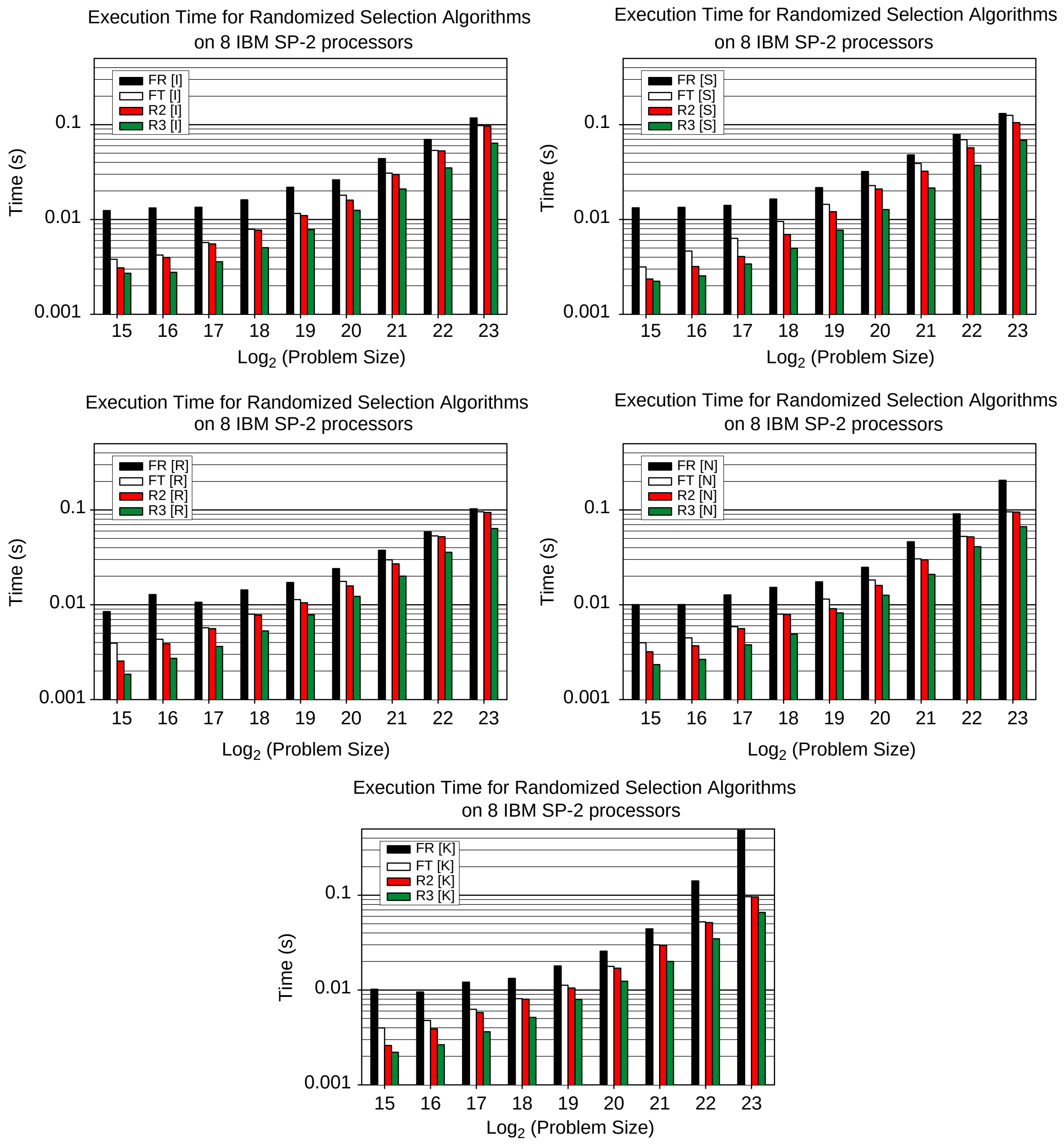


Fig. 2. Empirical performance of Fast versus UltraFast randomized selection algorithms with $p = 8$ nodes of an IBM SP-2-TN. Each of the five graphs represents one input class ([I], [S], [R], [N], and [K]) and increasing problem sizes on the x-axis. For each input class and problem size, there are four algorithms/variations reported: FR—the Fast randomized; FT—the modified Fast randomized with stopping criterion of $\max(p^2, 4096)$ instead of p^2; R2—the UltraFast randomized without the “Min/Max” selection improvement; and R3—the UltraFast randomized algorithm.

In addition, our UltraFast parallel, randomized selection algorithm, here designed and analyzed for a message-passing platform, would also be suitable for shared-memory multiprocessors (SMPs) and SMP Clusters [8]. In the SMP UltraFast selection algorithm, each communication step can be eliminated, simplified, or replaced with a shared-memory primitive. For instance, the SMP algorithm would be as follows. Each processor collects its portion of the sample from the corresponding block of the input and writes the sample to a shared-memory array. Thus, **Step 2**, the **Gather** communication, is eliminated. After a single processor determines the splitters k_1 and k_2 from the sample, the **Broadcast** communication in **Step 4** simplifies into a memory read by each processor. The **Combine** in **Step 6** may be replaced by the corresponding shared-memory primitive. The **Gather** in **Step 11** can be replaced with a shared-memory gather. We are currently investigating the performance of this SMP approach.

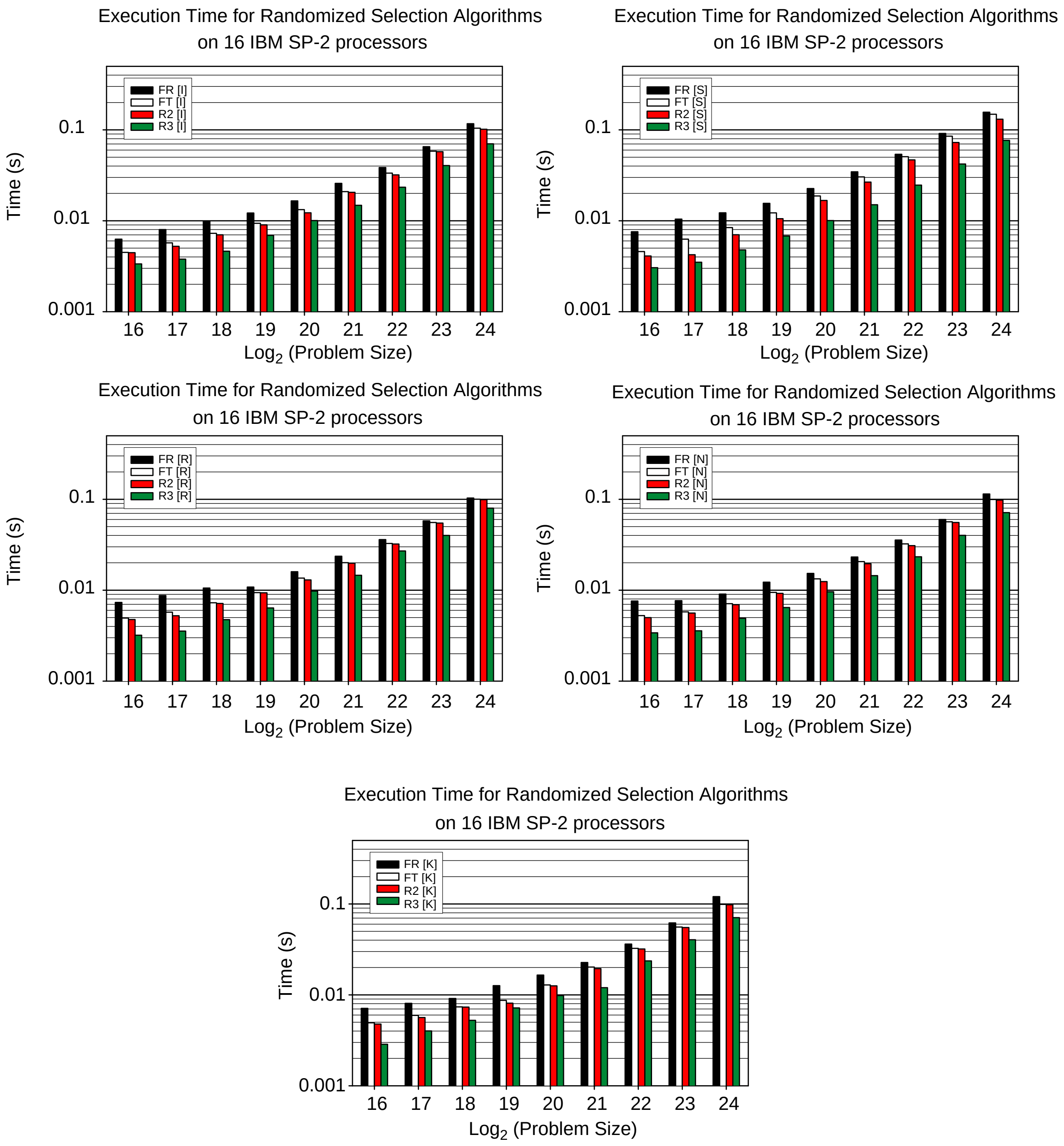


Fig. 3. Empirical performance of Fast versus UltraFast randomized selection algorithms with $p = 16$ nodes of an IBM SP-2-TN. Each of the five graphs represents one input class ([I], [S], [R], [N], and [K]) and increasing problem sizes on the x-axis. For each input class and problem size, there are four algorithms/variations reported: FR—the Fast randomized; FT—the modified Fast randomized with stopping criterion of $\max(p^2, 4096)$ instead of p^2; R2—the UltraFast randomized without the "Min/Max" selection improvement; and R3—the UltraFast randomized algorithm.

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