Structural Connectome Atlas Construction in the Space of Riemannian Metrics

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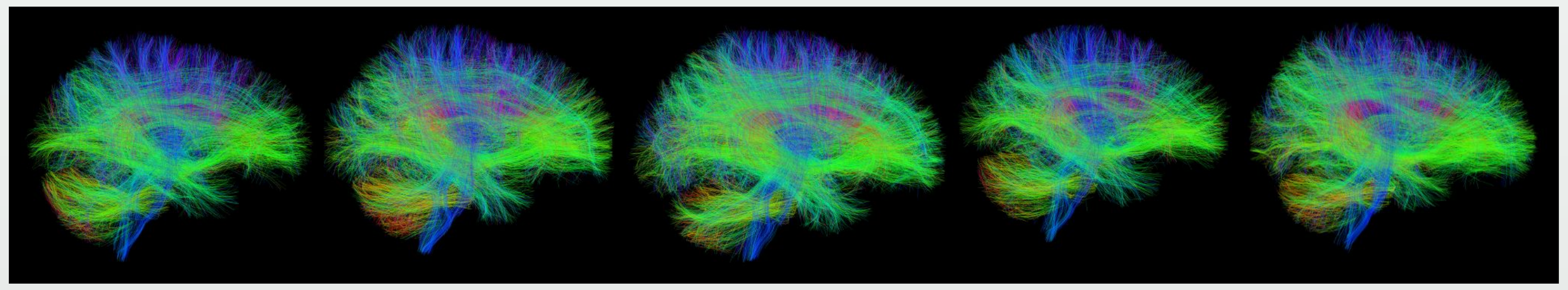
How to statistically analyze a population of connectomes?


[1]

- We propose a method of analyzing connectomes by representing tractography fibers as geodesics of a Riemannian metric, that is, as points on an infinite-dimensional manifold.
- We equip the Riemannian manifold with diffeomorphism-invariant Ebin metric to compute distances and geodesics between connectomes.
- We develop a Diffeomorphic Metric Registration framework to register connectomes
- We apply object-oriented statistical analysis to define an atlas as the Fréchet mean of a population of Riemannian metrics.

METHODS

1. Estimate a locally-invariant metric, $g_\alpha = e^{\alpha(x)}\tilde{g}$, conformal to the inverse-diffusion tensor metric, $\tilde{g} = D(x)^{-1}$, and whose geodesics match the tractography as defined by a vector field, V, by minimizing [2]:

$$F(\alpha) = \int_M ||\text{grad}\,\alpha - 2\nabla_V V||_{\tilde{g}}^2 dx$$

by solving $\Delta_{\tilde{g}}\alpha = 2\,\text{div}_{\tilde{g}}(\nabla_V V)$ for α.

2. Equip the infinite-dimensional space of all Riemannian metrics with a diffeomorphism-invariant Riemannian metric, called the Ebin metric [3], which is the integral of point-wise metrics on SPD:

$$G_g^E(h,k) = \int_M \text{Tr}\left(g^{-1}hg^{-1}k\right)\text{vol}(g),$$

where $h, k \in T_g\,\text{Met}(M), \text{vol}(g)$ the induced volume density of the metric g.

Distance is integral of point-wise distances on SPD:

$$\text{dist}_{\text{Met}}(g_0,g_1)^2 = \frac{16}{n}\int_M \left(a(x)^2 - 2a(x)b(x)\cos(\theta(x)) + b(x)^2\right)dx$$

Geodesic $g(x,t)$ between $g_0(x), g_1(x)$:

$$g(x,t) = \begin{cases} (q^2+r^2)^{\frac{2}{n}} g_0 \exp\left(\frac{\arctan(r/q)}{\kappa}k_0\right) & ,0<\kappa<\pi \\ q^{\frac{4}{n}} g_0 & ,\kappa = 0 \\ \left(1-\frac{\alpha+\beta}{\alpha}t\right)^{\frac{4}{n}} g_0 1_{[0,\frac{\alpha}{\alpha+\beta}]} + \left(\frac{\alpha+\beta}{\beta}t - \frac{\alpha}{\beta}\right)^{\frac{4}{n}} g_1 1_{[\frac{\alpha}{\alpha+\beta},1]} & ,\kappa \geq \pi \end{cases}$$

$$\alpha(x) = \sqrt[4]{\det(g_0(x))} \quad \beta(x) = \sqrt[4]{\det(g_1(x))}$$

$$k(x) = \log\left(g_0^{-1}(x)g_1(x)\right) \quad \theta(x) = \min\{\pi, \kappa(x)\}$$

$$k_0(x) = k(x) - \frac{\text{Tr}(k(x))}{n}\text{Id} \quad \kappa(x) = \frac{\sqrt{n\,\text{Tr}(k_0(x)^2)}}{4}$$

$$q(t,x) = 1 + t\left(\frac{\beta(x)\cos(\kappa(x)) - \alpha(x)}{\alpha(x)}\right)$$

$$r(t,x) = \frac{t\beta(x)\sin(\kappa(x))}{\alpha(x)}$$

3. Define "distance" of diffeomorphism φ to the identity, Ebin metric induces right-invariant distance on Diff(M):

$$\text{dist}_{\text{Diff}}^2(id,\varphi) = \text{dist}_{\text{Met}}^2(g,\varphi^* g)$$

4. Register two connectomes by finding φ that minimizes $E(\varphi)$:

$$E(\varphi) = \inf_{\varphi\in\text{Diff}(M)} \text{dist}_{\text{Diff}}^2(\text{id},\varphi) + \lambda\,\text{dist}_{\text{Met}}^2(g_0,\varphi^* g_1) \quad \text{(Algorithm 1)}$$

5. Build atlas, $\hat{g}$, of metrics $g_1,\ldots,g_N$ by finding g and $\varphi_1,\ldots,\varphi_N$ that minimize:

$$\hat{g} = \underset{g,\varphi_i}{\text{argmin}} \sum_{i=1}^{N} \text{dist}_{\text{Diff}}^2(\text{id},\varphi_i) + \lambda\,\text{dist}_{\text{Met}}^2(g,\varphi_i^* g_i) \quad \text{(Algorithm 2)}$$

V - vector field, div - Riemannian divergence, Δ*- Laplace-Beltrami operator, id - identity mapping, g - metric, E - energy function, n - metric dimension, * - pull back operation*

Algorithm 1 Inexact Metric Matching Algorithm

Inputs:
source and target metric g_0, g_1
Initialize:
learning rate ϵ; weight parameter λ; max iteration MaxIter
$\varphi, E \leftarrow \text{id}, 0$
for iteration $= 0$: MaxIter **do**
$\varphi^* g_1 \leftarrow (d\varphi)^T(g_1\circ\varphi)(d\varphi)$
$E \leftarrow \text{EbinEnergy}(\varphi^* g_1, g_0, \lambda)$
$v \leftarrow -\Delta^{-1}(\text{E.grad})$
$\psi \leftarrow \text{id} + \epsilon v$
$\varphi \leftarrow \psi\circ\varphi$
end for
return φ

Algorithm 2 Atlas Building Algorithm

Inputs:
sample metric fields list G
Initialize:
max iteration times MaxIter
for iteration $= 0$: MaxIter **do**
$g_{\text{mean}} \leftarrow \text{FréchetMean}(G)$
for $i = 0 : \text{len}(G)$ **do**
$\varphi \leftarrow \text{MetricMatching}(g_{\text{mean}}, G[i])$
$G[i] \leftarrow \varphi^* G[i]$
end for
end for
return g_{mean}

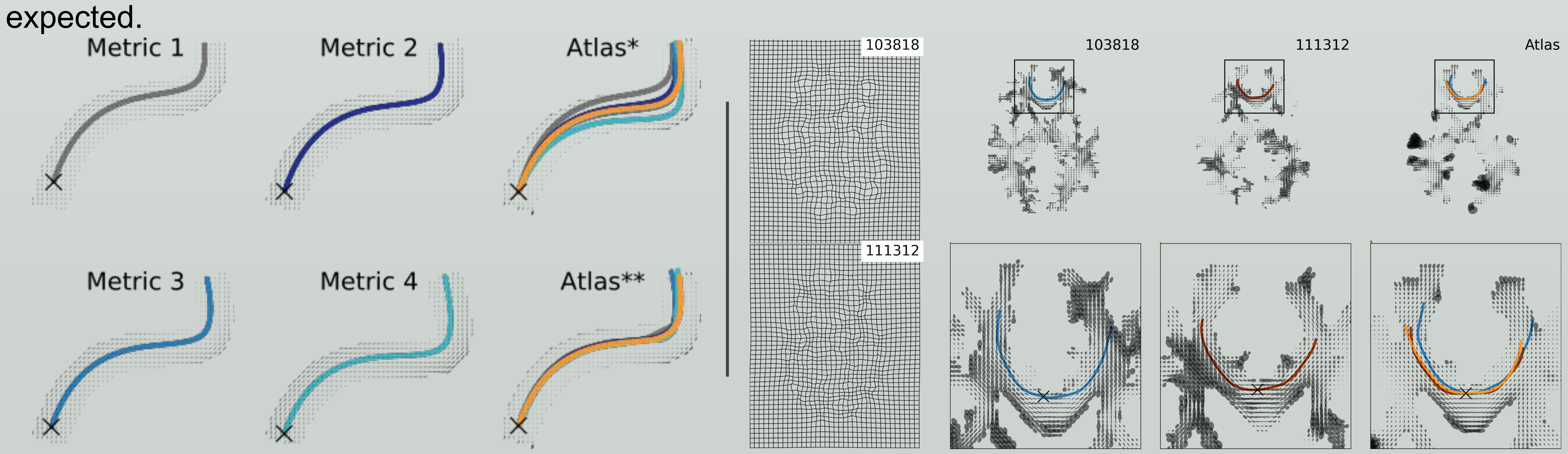
RESULTS

Hypothesis: The geodesic on atlas metric field should be the average of the geodesics on sample metric fields.

Data: Generated vector fields whose central integral curves are a family of parameterized cubic functions; a subset of subjects from the Human Connectome Project Young Adult (HCP) dataset.

Evaluation: We first estimated the adaptive metric conformal to the inverse-tensor metric. After finding the connectome metric for each subject, we ran atlas building Algorithm 2 to estimate the atlas shown in right column of each figure. Then, we map the geodesics of the individual connectome metrics to atlas space and compare with the atlas geodesic.

Conclusion: The atlas geodesic is nicely centered in the middle of the undeformed individual geodesics as expected.



orange: geodesic of altas metric; other colors: geodesics of sample metrics.
*undeformed geodesics of sample metrics. **geodesics of sample metrics deformed into atlas space.

REFERENCES

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