Homework 4: Expected value and variance

Due: Thursday, March 19 (Your answers are due at the beginning of the class on the due date. The preferred mode of submission is as a *PDF file* via Gradescope. Please see the instructions on the course webpage. These instructions will tell you how to create a Markdown document in R, and how to "knit" it into a PDF file.)

Note. As per course policy, discussing the concepts behind the problems is OK, but you must write up the answers entirely by yourself. Please see the course policy document (on Canvas or the course webpage) for details. **You need to show all the steps involved in obtaining the answer.**

Q1. ("Normalizing" a mass/density function) Let X be a random variable that takes values in $\{1, 2, 3, 4\}$, and suppose that the probability of taking the value i is $\alpha \cdot (i + 1)$, for $1 \leq i \leq 4$, for some α. What should α be so that this is a valid probability mass function?

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Q2. Let X be a random variable distributed uniformly in $[0, 2]$. (This is typically written as $X \sim Unif(0, 2)$.)

Compute the expected value of $X^3 + X^2$, i.e., $\mathbb{E}[X^3 + X^2]$.

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Q3. Suppose we toss a coin until we see a heads, and let X be the number of tosses. Recall that this is what we called the *binomial* distribution. Assume that it is a fair coin (equal probability of heads and tails).

(a) What is the p.m.f. of X? (I.e., for an integer i, what is $P(X = i)$?)

(b) What is $\mathbb{E}[X]$? ({*Hint:*} *this is a discrete variable that takes infinitely many values.)*

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Q4. (Computing variances) Consider three random variables:

X: distributed uniformly in $[-1, 1]$

Y: distributed uniformly in $[-4, 4]$

Z: distributed uniformly in $[4, 6]$

Compute the variance of each of the random variables. Can you comment on the relation between the variances of X and Z?

[*Hint:* In each case, compute the p.d.f. first, then the expected value and finally the variance.]

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Q5. Simulations to estimate the expectation Let X be a Gaussian random variable with mean 0 and variance 1 (i.e., $\mu = 0$ and $\sigma = 1$). Use R code to take 10k samples from X.

(a) plot the histogram and compare with the p.d.f. of X (using the formula from the textbook or Wikipedia). Show both plots.

(b) compute $\mathbb{E}[X^4]$ empirically (i.e., for each sample compute X^4 and take their average); now repeat this computation with 50k samples.