Homework 2: Total probability, Independence, Bayes Rule

Due: Thursday, February 13 (Your answers are due at the beginning of the class on the due date. The preferred mode of submission is as a *PDF file* via Gradescope. Please see the instructions on the course webpage. These instructions will tell you how to create a Markdown document in R, and how to "knit" it into a PDF file.)

Note. As per course policy, discussing the concepts behind the problems is OK, but you must write up the answers entirely by yourself. Please see the course policy document (on Canvas or the course webpage) for details. **You need to show all the steps involved in obtaining the answer.**

Q1. Answer the following questions on testing independence.

a. Suppose you roll a six-sided die. Are the following two events independent?

$$A = \text{"the number is (strictly) less than 5"}$$

$$B = \text{"the number is even"}$$

b. Suppose you flip two coins. Are the following events independent?

$$A : \text{"both flips have the same outcome (HH or TT)"}$$

$$B : \text{"at least one of the outcomes is T"}$$

Q2. A doctor sees a patient who may have a particular heart disease. According to the patient's family history and other risk factors, the doctor decides there is a 20% chance that the patient has the disease. The doctor then takes a blood test that turns out positive for the disease. However, there is a 10% chance that the test is positive when a patient does not have the disease. When a patient does have the disease, there is an 90% chance that the test will be positive. What is the probability that the patient has the disease given the positive test?

[*Hint:* Define clearly what the two events of interest are and apply Bayes rule. You may also want to recap the law of total probability.]

Q3. Let A and B be independent events and suppose that we have $P(A|B) = 1/2$ and $P(B|A \cup B) = 2/3$.

a. Suppose $P(B) = x$. Show that $P(A \cup B) = \frac{1+x}{2}$.
b. Find $P(B)$. [*Hint:* first show that $P(B|A \cup B) = P(B)/P(A \cup B)$.]

Q4. Solve problem 3.18 from the textbook. This relates to a point I mentioned a couple of times in class – independence and "disjointness" of events are very different things!

Q5. When we defined jointly independent random variables, we said that it is important to verify **all** the inequalities. The example below shows that having just the "product rule" does not suffice.

Suppose we pick a random integer i such that $1 \le i \le 8$, and define the following events:

$$A : i \text{ belongs to the set } \{1, 2, 3, 4\}$$

$$B : i \text{ belongs to the set } \{1, 2, 5, 6\}$$

$$C : i \text{ belongs to the set } \{1, 6, 7, 8\}$$

a. Prove that $P(A \cap B \cap C) = P(A)P(B)P(C)$.
b. Are A, B, C jointly independent? [*Hint:* can you find two of the events that are NOT independent?]