14. (d) **Stratified importance sampling:** We use the following random variable for generating the samples:

\[ X_i = 2\sqrt{\frac{i + \xi_i}{n}} \]

which has pdf \( kx \). Let’s determine \( k \):

\[ \int_{2\sqrt{\frac{i}{n}}}^{2\sqrt{\frac{i+1}{n}}} kx \, dx = k \frac{4}{2n} \quad \Rightarrow \quad k = \frac{n}{2} \]

Let’s derive its variance now:

\[ E(X_i) = \int_{2\sqrt{\frac{i}{n}}}^{2\sqrt{\frac{i+1}{n}}} \frac{n}{2} x^2 \, dx = \frac{n}{2} \left[ \frac{x^3}{3} \right]_{2\sqrt{\frac{i}{n}}}^{2\sqrt{\frac{i+1}{n}}} = \frac{4}{3\sqrt{n}} ((i + 1)^{3/2} - i^{3/2}) \]

We express the square of the mean as:

\[ E(X_i)^2 = \frac{16}{9n} ((i + 1)^3 + i^3 - 2(i(i + 1))^{3/2}) \]

\[ = \frac{16}{9n} (2(i^3 - (i(i + 1))^{3/2}) + 3i^2 + 3i + 1) \]

Also:

\[ E(X_i^2) = \int_{2\sqrt{\frac{i}{n}}}^{2\sqrt{\frac{i+1}{n}}} \frac{n}{2} x^3 \, dx = \frac{n}{2} \left[ \frac{x^4}{4} \right]_{2\sqrt{\frac{i}{n}}}^{2\sqrt{\frac{i+1}{n}}} = \frac{2}{n} (2i + 1) \]

We approximate the integral by creating \( Y \):

\[ Y = \frac{2}{n} \sum_{i=0}^{n-1} X_i \]

The variance of \( Y \) is obtained from the previous results:

\[ V(Y) = \frac{4}{n^2} \sum_{i=0}^{n-1} \frac{2}{n} (2i + 1) - \frac{16}{9n} (2(i^3 - (i(i + 1))^{3/2}) + 3i^2 + 3i + 1) \]

where

\[ \sum_{i=0}^{n-1} (2i + 1) = (n - 1)n + n = n^2 \]

and

\[ \sum_{i=0}^{n-1} (3i^2 + 3i + 1) = 3 \left( \frac{n - 1}{6} n (2n - 1) \right) + \frac{3(n - 1)n}{2} + n = n^3 \]

Thus:

\[ V(Y) = \frac{4}{n^2} \left[ 2n - \frac{16n^2}{9} - \frac{32}{9n} \sum_{i=0}^{n-1} i^3 - (i(i + 1))^{3/2} \right] = \frac{8}{n} - \frac{64}{9} + \frac{128}{9n^3} \sum_{i=0}^{n-1} (i(i + 1))^{3/2} - i^3 = \]

\[ = \frac{8}{n} - \frac{128}{9} \left[ \frac{1}{2} - \sum_{i=0}^{n-1} \left( \frac{i + 1}{n} \right)^{3/2} - \left( \frac{i}{n} \right)^3 \right] \]
Clearly, that sum is pretty ugly, but we can see that it is the approximation of the following integral for \( f(x) = x^{3/2} \):

\[
\int_0^1 f(x) f(x) \, dx = \frac{3}{2} \int_0^1 x^{1/2} x^{3/2} \, dx = \frac{3}{2} \left[ \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^1 = \frac{1}{2}
\]

I could not express the error of the integral sum as a function of \( n \). However, we can still compute it (that’s why we have computers :).

Figure 1: Error of stratified importance sampling and standard deviation prediction.

Interesting to see how the sum cancelled out the \( O(1/n) \) term, and made \( O(1/n^3) \) dominant.