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1. **INTRODUCTION**

SABR model is a CEV model augmented by stochastic volatility that assumes the forward rate evolves under the associated forward (terminal) measure $\mathbb{Q}^T$

$$dF_{t,T} = \alpha_t F_{t,T}^\beta dW_t, \quad d\alpha_t = \nu\alpha_t dZ_t, \quad dW_t dZ_t = \rho dt \quad (1)$$

for time $t$ between initial time $s$ and maturity $T$. The $F_{t,T}$ is a forward rate process with initial value $F_{s,T} = f$. The $\alpha_t$ is the stochastic volatility with initial value $\alpha_s = a$. The parameter $a$ cannot be observed from the market, however it can be derived analytically from the at-the-money implied volatility as we shall see in due course. The factor $\nu$ is known as the volatility of volatility, which adjusts the degree of volatility clustering in time. The parameter $\beta \in [0,1]$ controls the relationship between the forward rate and the at-the-money volatility. A $\beta < 1$ (“non-lognormal” case) leads to skews in the implied volatilities.

In the case of $\beta \approx 1$, if the market were to move up or down, the level of the at-the-money volatility would not be significantly affected, whereas when $\beta < 1$ the volatility increases as forward rate falls (i.e. volatility and forward move in opposite direction). The closer to 0 the more pronounced would be this effect. The correlation parameter $\rho$ plays a similar role as the $\beta$ does. It defines how the market moves in sync with the volatility dynamics. The model parameters $\nu$, $a$, $\beta$ and $\rho$ are all assumed to be deterministic and time invariant.

2. **ASYMPTOTIC SOLUTION BY HAGAN ET AL.**

Using singular perturbation techniques, Hagan et al. [1] provide a closed form asymptotic solution (up to the accuracy of a series expansion) for prices of vanilla instruments. The value of a vanilla option under the SABR model is given by the appropriate Black formula provided that the correct Black implied volatility is used. Given the forward initial value $F_{s,T} = f$ and the expiry time $\tau = T - s$, the Black implied volatility can be derived as a function of strike price $K$ from a given set of SABR parameters $\nu$, $a$, $\beta$ and $\rho$ by
\[
\sigma_K = \frac{\zeta}{\chi} \cdot a \left( 1 + \left( \frac{(1 - \beta)^2 a^2}{24p^2} + \frac{\rho \beta \nu a}{4p} + \frac{(2 - 3 \rho^2)v^2}{24} \right) \tau \right)
\]
\[
p \left( 1 + \left( \frac{(1 - \beta)^2 q^2}{24} + \frac{(1 - \beta)^4 q^4}{1920} \right) \right)
\]

where \( p = (fK)^{1-\beta} \), \( q = \ln \frac{f}{K} \), \( \zeta = \frac{\nu pq}{a} \), \( \chi = \ln \left( \frac{\sqrt{1 - 2\rho \zeta + \zeta^2 + \zeta - \rho}}{1 - \rho} \right) \)

When \( K = f \), equation (2) reduces to give the at-the-money implied volatility

\[
\sigma_f = \frac{a}{f^{1-\beta}} \left( 1 + \left( \frac{(1 - \beta)^2 a^2}{24p^2} + \frac{\rho \beta \nu a}{4p} + \frac{(2 - 3 \rho^2)v^2}{24} \right) \tau \right)
\]

The (3) shows that there exists a relationship

\[
\ln \sigma_f = \ln a - (1 - \beta) \ln f + \ ...
\]

It indicates that the value of \( \beta \) can be estimated from a log-log regression of \( \sigma_f \) and \( f \) with historical data by ignoring terms involving \( \tau \). Alternatively, since the parameters \( \beta \) and \( \rho \) in SABR model control the distribution function in similar ways (i.e. both control the skewness of the distribution), the redundancy between the two parameters allows one to calibrate the model by fixing \( \beta \) to an assumption (e.g. \( \beta = 0.5 \)). The decision is often made on the basis of market experience. The remaining parameters \( a \), \( \nu \) and \( \rho \) have different effects on the volatility curve. The parameter \( a \) mainly controls the overall magnitude of the curve, the \( \nu \) controls how much smile (i.e. convexity) the curve exhibits and the \( \rho \) controls the curve’s skew.

As shown in (3) the parameter \( a \) has a functional form with the at-the-money volatility \( \sigma_f \).

Inverting the equation gives the value of \( a \) as a root of a cubic equation if the \( \nu \) and \( \rho \) are known (in general, the smallest positive root would be taken if there were three real roots)

\[
\frac{(1 - \beta)^2 \tau}{24f^2(1-\beta)} a^3 + \frac{\rho \beta \nu \tau}{4f^{1-\beta}} a^2 + \left( 1 + \frac{(2 - 3 \rho^2)v^2 \tau}{24} \right) a - \sigma_f f^{1-\beta} = 0
\]

This indicates that in SABR model we only need to calibrate \( \rho \) and \( \nu \) to implied volatility surface, providing that the value of \( \beta \) is prescribed and the at-the-money implied volatility \( \sigma_f \) is given. The
calibration is performed at each maturity of the volatility surface by minimizing the objective function defined as a sum of squared residuals (or sum of vega weighted squared residuals)

$$\argmin_{\nu, \rho} \sum_{i=1}^{N} \left( \sigma_{K_i}^{mkt} - \sigma_{K_i}^{SABR|f,v,a,\beta,\rho} \right)^2$$

Various nonlinear optimization routines can be used to carry out the calibration, for example, Levenberg-Marquardt method or Nelder-Mead simplex method.

3. **Correction to Hagan et al. Solution**

The $\zeta$ in (2) was defined in equation (A.57c) in [1]. When assuming CEV model for the forward process (i.e. $C(F) = F^\beta$), it has the form

$$\zeta = \frac{\nu}{a} \int_K f \frac{dF}{C(F)} = \frac{\nu}{a} \int_K f \frac{du}{u^\beta} = \frac{\nu}{a} \frac{f^{1-\beta} - K^{1-\beta}}{1-\beta} = \frac{\nu}{a} \eta, \quad \eta = \frac{f^{1-\beta} - K^{1-\beta}}{1-\beta}$$

By expanding, we have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^y - e^{-y} = 2y \left( 1 + \frac{y^2}{3!} + \frac{y^4}{5!} + \cdots \right)$$

$$\Rightarrow p \left( e^{\frac{1-\beta}{2} q} - e^{-\frac{1-\beta}{2} q} \right) = p(1 - \beta)q \left( 1 + \frac{(1 - \beta)^2 q^2}{24} + \frac{(1 - \beta)^4 q^4}{1920} + \cdots \right),$$

for $y = \frac{1 - \beta}{2} q$, $p = (fK)^{\frac{1-\beta}{2}}$, $q = \ln \frac{f}{K}$

$$\Rightarrow f^{1-\beta} - K^{1-\beta} = (1 - \beta)pq \left( 1 + \frac{(1 - \beta)^2 q^2}{24} + \frac{(1 - \beta)^4 q^4}{1920} + \cdots \right)$$

$$\Rightarrow \eta = pq \left( 1 + \frac{(1 - \beta)^2 q^2}{24} + \frac{(1 - \beta)^4 q^4}{1920} + \cdots \right)$$

Thus $\zeta$ can be written as

$$\zeta = \frac{\nu}{a} pq \left( 1 + \frac{(1 - \beta)^2 q^2}{24} + \frac{(1 - \beta)^4 q^4}{1920} + \cdots \right)$$

Clearly the expression of $\zeta$ used in (2) is just an approximation of (9) truncating all higher order terms of $q$. This leads to a correction to the original Hagan et al. solution proposed by Obloj [2] in 2008, where he
uses $\zeta = \frac{\nu}{a} \eta$ in (7) to calculate $\zeta$ rather than $\zeta = \frac{\nu}{a} pq$ in (2). Plugging the $\zeta$ in (9) into the original Hagan et al. solution (2), we get the improved implied volatility formula

$$
\sigma_K = \frac{\nu q}{\chi} \left( 1 + \left( \frac{(1 - \beta)^2 a^2}{24p^2} + \frac{\rho \beta \nu a}{4p} + \frac{(2 - 3 \rho^2) \nu^2}{24} \right) \tau \right) \quad \text{where}
$$

$$
p = (fK)^{1-\beta}, \quad q = \ln \frac{f}{K}, \quad \zeta = \frac{\nu f^{1-\beta} - K^{1-\beta}}{1 - \beta}, \quad \chi = \ln \left( \frac{1 - 2 \rho \zeta + \zeta^2 + \zeta - \rho}{1 - \rho} \right) \quad (10)
$$

Two special cases must be addressed. First when $K \to f$, we get the at-the-money volatility

$$
\lim_{K \to f} \sigma_K = \frac{a}{f^{1-\beta}} \left( 1 + \left( \frac{(1 - \beta)^2 a^2}{24f^{2(1-\beta)}} + \frac{\rho \beta \nu a}{4f^{1-\beta}} + \frac{(2 - 3 \rho^2) \nu^2}{24} \right) \tau \right) \quad (11)
$$

knowing that (or simply derived from the series expansion in (8))

$$
\lim_{K \to f} \frac{f^{1-\beta} - K^{1-\beta}}{(1 - \beta)q} = \lim_{K \to f} \frac{-(1 - \beta)K^{-\beta}}{-(1 - \beta) \frac{1}{K}} = f^{1-\beta}, \quad \lim_{K \to f} \frac{\zeta}{\chi} = 1 \quad (12)
$$

$$
\Rightarrow \lim_{K \to f} \frac{\nu q}{\chi} = \lim_{K \to f} \frac{a \zeta}{\chi} \frac{(1 - \beta)q}{f^{1-\beta} - K^{1-\beta}} = \frac{a}{f^{1-\beta}}
$$

Second when $\beta \to 1$, we have the implied volatility in lognormal SABR

$$
\lim_{\beta \to 1} \sigma_K = \frac{\nu q}{\chi} \left( 1 + \left( \frac{\rho \beta \nu a}{4} + \frac{(2 - 3 \rho^2) \nu^2}{24} \right) \tau \right) \quad (13)
$$

using the fact that

$$
\lim_{\beta \to 1} \frac{f^{1-\beta} - K^{1-\beta}}{1 - \beta} = \lim_{\beta \to 1} \frac{-f^{1-\beta} \ln f + K^{1-\beta} \ln K}{-1} = \ln \frac{f}{K} = q \Rightarrow \lim_{\beta \to 1} \zeta = \frac{\nu q}{a} \quad (14)
$$
REFERENCES
