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The note is prepared for the purpose of summarizing local volatility models frequently encountered in derivative pricing. It at first derives the Kolmogorov forward and backward equations, which fundamentally govern the transition probability density of the diffusion process in derivative price dynamics. Subsequently, it introduces the local volatility model in the context of Dupire formula and then presents a PDE based local volatility model, in which the local volatility function is parametrized to be piecewise linear in log-moneyness and piecewise constant in time.

1. **Kolmogorov Forward and Backward Equations**

The time evolution of the transition probability density function is governed by Kolmogorov forward and backward equations, which will be introduced as follows, without loss of generality, in multi-dimension.

1.1. Kolmogorov Forward Equation

Let’s consider the following $m$-dimensional stochastic spot process $S_t \in \mathbb{R}^m$ driven by an $n$-dimensional Brownian motion $W_t$ whose correlation matrix $\rho$ is given by $\rho dt = dW_t dW_t'$

$$dS_t = A(t, S_t) dt + B(t, S_t) dW_t$$

We derive the dynamics of $h$, where $h: \mathbb{R}^m \rightarrow \mathbb{R}$ in this case is a scalar-valued Borel-measurable function only on variable $S_t$

$$dh(S_t) = J_h dS_t + \frac{1}{2} dS_t' H_h dS_t = J_h Adt + J_h BdW_t + \frac{1}{2} dW_t' B' H_h B dW_t$$

(2)

where $J_h$ is the $1 \times m$ Jacobian (i.e. the same as gradient if $h$ is a scalar-valued function) and $H_h$ the $m \times m$ Hessian (with subscripts of $S$ now denoting the indices of vector components)

$$[J_h]_i = \frac{\partial h}{\partial S_i} \quad \text{and} \quad [H_h]_{ij} = \frac{\partial^2 h}{\partial S_i \partial S_j}$$

(3)

Expanding the expression in (2), we have
\[
dh = \sum_{i=1}^{m} \frac{\partial h}{\partial S_i} A_i \, dt + \sum_{i=1}^{m} \frac{\partial h}{\partial S_i} \sum_{k=1}^{n} B_{ik} \, dW_k + \frac{1}{2} \sum_{i,j=1}^{m} \frac{\partial^2 h}{\partial S_i \partial S_j} \sum_{k=1}^{n} B_{ik} \rho_{ij} B_{jk} \, dt
\]

\[
= \left( \sum_{i=1}^{m} \frac{\partial h}{\partial S_i} A_i + \frac{1}{2} \sum_{i,j=1}^{m} \frac{\partial^2 h}{\partial S_i \partial S_j} \Sigma_{ij} \right) dt + \sum_{i=1}^{m} \frac{\partial h}{\partial S_i} \sum_{k=1}^{n} B_{ik} \, dW_k
\]

where \( \Sigma = B \rho B' \) is the \( m \times m \) instantaneous variance-covariance matrix of \( dS \). Integrating on both sides of (4) from \( t \) to \( T \), we have

\[
h(S_T) - h(S_t) = \int_t^T \left( \sum_{i=1}^{m} \frac{\partial h}{\partial S_i} A_i + \frac{1}{2} \sum_{i,j=1}^{m} \frac{\partial^2 h}{\partial S_i \partial S_j} \Sigma_{ij} \right) \, du + \int_t^T \sum_{i=1}^{m} \frac{\partial h}{\partial S_i} \sum_{k=1}^{n} B_{ik} \, dW_k
\]

(5)

Taking expectation on both sides of (5), we get

\[
\text{LHS} = \mathbb{E}_t [h(S_T)] - h(S_t) = \int_\Omega h_y p_{T,y|t,x} \, dy - h_x
\]

\[
\text{RHS} = \mathbb{E}_t \left[ \int_t^T \left( \sum_{i=1}^{m} \frac{\partial h}{\partial S_i} A_i + \frac{1}{2} \sum_{i,j=1}^{m} \frac{\partial^2 h}{\partial S_i \partial S_j} \Sigma_{ij} \right) \, du \right] + \mathbb{E}_t \left[ \int_t^T \sum_{i=1}^{m} \frac{\partial h}{\partial S_i} \sum_{k=1}^{n} B_{ik} \, dW_k \right]
\]

(6)

\[
= \int_t^T \sum_{i=1}^{m} \mathbb{E}_t \left[ \frac{\partial h}{\partial S_i} A_i \right] \, du + \frac{1}{2} \int_t^T \sum_{i,j=1}^{m} \mathbb{E}_t \left[ \frac{\partial^2 h}{\partial S_i \partial S_j} \Sigma_{ij} \right] \, du
\]

where \( p_{T,y|t,x} \) is the transition probability density having \( S_T = y \) at \( T \) given \( S_t = x \) at \( t \) (i.e. if we solve the equation (1) with the initial condition \( S_t = x \in \mathbb{R}^m \), then the random variable \( S_T = y \in \Omega \) has a density \( p_{T,y|t,x} \) in the \( y \) variable at time \( T \)). Differentiating (6) with respect to \( T \) on both sides, we have

\[
\int_\Omega h_y \frac{\partial p_{T,y|t,x}}{\partial T} \, dy = \sum_{i=1}^{m} \mathbb{E}_t \left[ \frac{\partial h}{\partial S_i} A_i \right] + \frac{1}{2} \sum_{i,j=1}^{m} \mathbb{E}_t \left[ \frac{\partial^2 h}{\partial S_i \partial S_j} \Sigma_{ij} \right]
\]

(7)

\[
= \sum_{i=1}^{m} \int_\Omega \frac{\partial h_y}{\partial y_i} A_i p_{T,y|t,x} \, dy + \frac{1}{2} \sum_{i,j=1}^{m} \int_\Omega \frac{\partial^2 h_y}{\partial y_i \partial y_j} \Sigma_{ij} p_{T,y|t,x} \, dy
\]
If we assume $\Omega \equiv \mathbb{R}^m$ and also assume the probability density $p$ and its first derivatives $\partial p / \partial y_i$ vanish at a higher order of rate than $h$ and $\partial h / \partial y_i$ as $y_i \to \pm \infty \ \forall \ i = 1, \cdots, m$, then we can integrate by parts for the right hand side of (7), once for the first integral and twice for the second

\[
\int_{\Omega} \frac{\partial h_y}{\partial y_i} A_i p \, dy = \int_{\Omega} \left( \frac{h_y A_i p}{y_i = -\infty} \right)_{y_i = \infty} dy_i - \int_{\Omega} h_y \frac{\partial (A_i p)}{\partial y_i} dy \quad \text{and}
\]

\[
\int_{\Omega} \frac{\partial^2 h_y}{\partial y_i \partial y_j} \Sigma_{ij} p \, dy = \int_{\Omega} \left( \frac{\partial h_y \Sigma_{ij} p}{y_i = -\infty} \right)_{y_j = \infty} dy_j - \int_{\Omega} \frac{\partial h_y \partial (\Sigma_{ij} p)}{\partial y_j} dy_j + \int_{\Omega} \frac{\partial^2 (\Sigma_{ij} p)}{\partial y_i \partial y_j} dy_j
\]

(8)

where \[ \int_{\Omega} (\cdot) dy_i = \int_{\Omega} \cdots \int_{\Omega} \cdots \int_{\Omega} (\cdot) dy_1 \cdots dy_{i-1} dy_i dy_{i+1} \cdots dy_m \]

Plugging the results of (8) into (7), we have

\[
\int_{\Omega} h_y \frac{\partial p}{\partial T} dy = - \sum_{i=1}^{m} \int_{\Omega} h_y \frac{\partial (A_i p)}{\partial y_i} dy + \frac{1}{2} \sum_{i,j=1}^{m} \int_{\Omega} h_y \frac{\partial^2 (\Sigma_{ij} p)}{\partial y_i \partial y_j} dy
\]

\[
\Rightarrow \int_{\Omega} h_y \left( \frac{\partial p}{\partial T} + \sum_{i=1}^{m} \frac{\partial (A_i p)}{\partial y_i} - \frac{1}{2} \sum_{i,j=1}^{m} \frac{\partial^2 (\Sigma_{ij} p)}{\partial y_i \partial y_j} \right) dy = 0
\]

(9)

By the arbitrariness of $h$, we conclude that for any $y \in \Omega$

\[
\frac{\partial p}{\partial T} + \sum_{i=1}^{m} \frac{\partial (A_i p)}{\partial y_i} - \frac{1}{2} \sum_{i,j=1}^{m} \frac{\partial^2 (\Sigma_{ij} p)}{\partial y_i \partial y_j} = 0, \quad \Sigma = B \rho B'
\]

(10)

This is the Multi-dimensional Fokker-Planck Equation (a.k.a. Kolmogorov Forward Equation) [1]. In this equation, the $t$ and $x$ are held constant, while the $T$ and $y$ are variables (called “forward variables”).

In the one-dimensional case, it reduces to

\[
\frac{\partial p}{\partial T} + \frac{\partial (Ap)}{\partial y} - \frac{1}{2} \frac{\partial^2 (B^2 p)}{\partial y^2} = 0
\]

(11)

where $A = A(T, y)$ and $B = B(T, y)$ are then scalar functions.
1.2. Kolmogorov Backward Equation

Let’s write conditional expectation of $h(S_t)$ by

$$g(t, S_t) = \mathbb{E}[h(S_T)|\mathcal{F}_t] = \mathbb{E}_t[h(S_T)]$$

Because for $s \leq t \leq T$ we have

$$g(s, S_s) = \mathbb{E}_s[h(S_T)] = \mathbb{E}_s[\mathbb{E}_t[h(S_T)]] = \mathbb{E}_s[g(t, S_t)]$$

the $g(t, S_t)$ is a martingale by the tower rule (i.e. If $\mathcal{H}$ holds less information than $\mathcal{G}$, then $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}]$). The dynamics of the $g(t, S_t)$ is given by

$$dg = \frac{\partial g}{\partial t} dt + J_g dS_t + \frac{1}{2} dS_t' H_g dS_t$$

$$= \frac{\partial g}{\partial t} dt + J_g A dt + J_g B dW_t + \frac{1}{2} dW_t' B' H_g B dW_t$$

where $J_g$ is the Jacobian (i.e. the same as gradient if $g$ is a scalar-valued function) and $H_g$ the Hessian of $g$ with respect to $S$ (with subscripts denoting the indices of vector components)

$$J_g = \left( \frac{\partial g}{\partial S_1} \ldots \frac{\partial g}{\partial S_m} \right), \quad H_g = \begin{pmatrix} \frac{\partial^2 g}{\partial S_1^2} & \ldots & \frac{\partial^2 g}{\partial S_1 \partial S_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 g}{\partial S_m \partial S_1} & \ldots & \frac{\partial^2 g}{\partial S_m^2} \end{pmatrix}$$

Expanding (14), we have

$$dg = \left( \frac{\partial g}{\partial t} + \sum_{i=1}^m \frac{\partial g}{\partial S_i} A_i + \frac{1}{2} \sum_{i,j=1}^m \frac{\partial^2 g}{\partial S_i \partial S_j} \Sigma_{ij} \right) dt + \sum_{i=1}^m \frac{\partial g}{\partial S_i} \sum_{k=1}^n B_{ik} dW_k$$

Since $g(t, S_t)$ is a martingale, the $dt$-term must vanish, which gives

$$\frac{\partial g}{\partial t} + \sum_{i=1}^m \frac{\partial g}{\partial S_i} A_i + \frac{1}{2} \sum_{i,j=1}^m \frac{\partial^2 g}{\partial S_i \partial S_j} \Sigma_{ij} = 0$$

This is the multi-dimensional Feynman-Kac formula.

Using the transition probability density $p_{T,y|t,x}$, we can write the expectation as
\( g_{t,x} = \mathbb{E}_t[h(S_T)] = \int_\Omega h_y p_{T,y|t,x} dy \) \hspace{1cm} (18)

The formula (17) defines that

\[
\frac{\partial}{\partial t} \int_\Omega h_y p dy + \sum_{i=1}^{m} A_i \frac{\partial}{\partial x_i} \int_\Omega h_y p dy + \frac{1}{2} \sum_{i,j=1}^{m} \Sigma_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \int_\Omega h_y p dy = 0
\]

\[
\Rightarrow \int_\Omega h_y \left( \frac{\partial p}{\partial t} + \sum_{i=1}^{m} A_i \frac{\partial p}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{m} \Sigma_{ij} \frac{\partial^2 p}{\partial x_i \partial x_j} \right) dy = 0
\]

(19)

By the arbitrariness of \( h \), we have

\[
\frac{\partial p}{\partial t} + \sum_{i=1}^{m} A_i \frac{\partial p}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{m} \Sigma_{ij} \frac{\partial^2 p}{\partial x_i \partial x_j} = 0, \quad \Sigma = B \rho B'
\]

(20)

This is the multi-dimensional Kolmogorov Backward Equation. In this equation, the \( T \) and \( y \) are held constant, while the \( t \) and \( x \) are variables (called “backward variables”). In the 1-D case, it reduces to

\[
\frac{\partial p}{\partial t} + A \frac{\partial p}{\partial x} + \frac{1}{2} B^2 \frac{\partial^2 p}{\partial x^2} = 0
\]

(21)

where \( A = A(t,x) \) and \( B = B(t,x) \) are then scalar functions.
2. LOCAL VOLATILITY

In local volatility models, the volatility process is assumed to be a function of both the spot level and the time. It is one step generalization of the well-known Black-Scholes model. Under risk neutral measure, the spot process (e.g. an equity or an FX rate) is assumed to follow a geometric Brownian motion

\[
\frac{dS_t}{S_t} = \mu_t dt + \sigma(t, S_t) d\tilde{W}_t, \quad \mu_t = r_t - q_t
\]

with cash rate \(r_t\) and dividend rate \(q_t\) (or foreign cash rate for FX).

2.1. Local Volatility by Vanilla Call

Under the assumption of deterministic \(r_t\), the European (vanilla) call option price can be expressed as a function of maturity time \(T\) and strike \(K\)

\[
C_{T,K|t,x} = \mathbb{E}_t[D_{t,T}(S_T - K)^+] = D_{t,T} \int_0^\infty (y - K)p_{T,y|t,x} dy
\]

where \(D_{t,T} = \exp\left(-\int_t^T r_u du\right)\) is the deterministic discount factor and \(p_{T,y|t,x}\) is the transition probability density having spot \(S_T = y\) at \(T\) given initial condition \(S_t = x\) at \(t\). Differentiating (23) with respect to \(K\), we have the first order and second order partial derivative

\[
\frac{\partial C_{T,K|t,x}}{\partial K} = -D_{t,T} \int_K^\infty p_{T,y|t,x} dy , \quad \frac{\partial^2 C_{T,K|t,x}}{\partial K^2} = D_{t,T} p_{T,K|t,x}
\]

which gives the transition probability density function by

\[
p_{T,K|t,x} = \frac{1}{D_{t,T}} \frac{\partial^2 C_{T,K|t,x}}{\partial K^2}
\]

The (25) is also known as Breeden-Litzenberger formula.

Taking the first derivative of \(C_{T,K|t,x}\) in (23) with respect to \(T\), we find

\[
\frac{\partial C_{T,K|t,x}}{\partial T} = -r_T C_{T,K|t,x} + D_{t,T} \int_K^\infty (y - K) \frac{\partial p_{T,y|t,x}}{\partial T} dy
\]
\[-r_T C_{T,K|t,x} + D_{t,T} \int_K^{\infty} (y - K) \left( \frac{1}{2} \frac{\partial^2 (\sigma^2_T y^2 p_{T,y|t,x})}{\partial y^2} - \frac{\partial (\mu_T y p_{T,y|t,x})}{\partial y} \right) dy \]

using the Kolmogorov Forward Equation (11)

\[
\frac{\partial p_{T,y|t,x}}{\partial T} = \frac{1}{2} \frac{\partial^2 (\sigma^2_T y^2 p_{T,y|t,x})}{\partial y^2} - \frac{\partial (\mu_T y p_{T,y|t,x})}{\partial y}
\]

Applying integration by parts to the integrals on the right hand side of (26) yields

\[
\int_K^{\infty} (y - K) \left( \frac{\partial (\mu_T y p_{T,y|t,x})}{\partial y} \right) dy = (y - K)\mu_T y p_{T,y|t,x} \bigg|_{y=K}^{\infty} - \int_K^{\infty} \mu_T y p_{T,y|t,x} dy
\]

\[
= -\mu_T \left( \int_K^{\infty} (y - K) p_{T,y|t,x} dy + K \int_K^{\infty} p_{T,y|t,x} dy \right) = \frac{\mu_T K}{D_{t,T}} \frac{\partial C_{T,K|t,x}}{\partial K} - \frac{\mu_T C_{T,K|t,x}}{D_{t,T}}
\]

and

\[
\int_K^{\infty} (y - K) \frac{\partial^2 (\sigma^2_T y^2 p_{T,y|t,x})}{\partial y^2} dy
\]

\[
= (y - K) \left( \frac{\partial (\sigma^2_T y^2 p_{T,y|t,x})}{\partial y} \right) \bigg|_{y=K}^{\infty} - \int_K^{\infty} \frac{\partial (\sigma^2_T y^2 p_{T,y|t,x})}{\partial y} dy
\]

\[
= \sigma^2_T y^2 p_{T,y|t,x} \bigg|_{y=K}^{\infty} = \sigma^2_T K^2 p_{T,K|t,x} = \frac{\sigma^2_{T,K} K^2}{D_{t,T}} \frac{\partial^2 C_{T,K|t,x}}{\partial K^2}
\]

where we have \(p_{T,\infty|t,x} = 0\) and \(\partial p_{T,\infty|t,x} / \partial y = 0\) assuming the probability density \(p_{T,y|t,x}\) and its first derivative vanish at a higher order of rate as \(y \to \infty\). Plugging (28) into (26), we find

\[
\frac{\partial C_{T,K|t,x}}{\partial T} = -r_T C_{T,K|t,x} + \frac{\sigma^2_{T,K} K^2}{2} \frac{\partial^2 C_{T,K|t,x}}{\partial K^2} - \mu_T K \frac{\partial C_{T,K|t,x}}{\partial K} + \mu_T C_{T,K|t,x}
\]

\[
\frac{\sigma^2_{T,K} K^2}{2} \frac{\partial^2 C_{T,K|t,x}}{\partial K^2} - \mu_T K \frac{\partial C_{T,K|t,x}}{\partial K} - q_T C_{T,K|t,x}
\]

and eventually the Dupire formula for the local volatility \(\sigma_{T,K}\) expressed in terms of vanilla call price

\[
\frac{\sigma^2_{T,K}}{2} = \frac{\partial C_{T,K|t,x}}{\partial T} + \mu_T K \frac{\partial C_{T,K|t,x}}{\partial K} + q_T C_{T,K|t,x}
\]

\[
K^2 \frac{\partial^2 C_{T,K|t,x}}{\partial K^2}
\]
2.2. Local Volatility by Forward Call

Sometimes, it is more convenient to express the Dupire formula in terms of a forward (i.e., undiscounted) call, which is defined as

\[ C_{T,K|t,x} = \mathbb{E}_t[(S_T - K)^+] = \int_K^\infty (y - K)p_{T,y|t,x}dy = \frac{C_{T,K|t,x}}{D_{t,T}} \]  \hspace{1cm} (31)

with \( C_{T,K|t,x} \) given in (23) (note that (31) is true only if the interest rate is deterministic). Following a similar derivation, we find that

\[ \frac{\partial C_{T,K|t,x}}{\partial K} = -\int_K^\infty y p_{T,y|t,x}dy, \quad \frac{\partial^2 C_{T,K|t,x}}{\partial K^2} = p_{T,K|t,x} \quad \text{and} \]
\[ \frac{\partial C_{T,K|t,x}}{\partial T} = \int_K^\infty (y - K) \frac{\partial p_{T,y|t,x}}{\partial T}dy = \frac{1}{2} \sigma_T^2 K^2 \frac{\partial^2 C_{T,K|t,x}}{\partial K^2} + \mu_T (C_{T,K|t,x} - K \frac{\partial C_{T,K|t,x}}{\partial K}) \]  \hspace{1cm} (32)

Therefore the Dupire formula for \( \sigma_{T,K} \) expressed in terms of forward call price reads

\[ \frac{\sigma_{T,K}^2}{2} = \frac{\frac{\partial C_{T,K|t,x}}{\partial T} + \mu_T \frac{\partial C_{T,K|t,x}}{\partial K} - \mu_T C_{T,K|t,x}}{K^2 \frac{\partial^2 C_{T,K|t,x}}{\partial K^2}} \]  \hspace{1cm} (33)

2.2.1. Local Variance as a Conditional Expectation of Instantaneous Variance

The forward call (31) can be expressed as

\[ C_{T,K|t,x} = \mathbb{E}_t[(S_T - K)^+] = \mathbb{E}[\mathcal{H}(S_T - K)(S_T - K)] = \mathbb{E}[\mathcal{H}(S_T - K)S_T] - K \mathbb{E}[\mathcal{H}(S_T - K)] \]  \hspace{1cm} (34)

where \( \mathcal{H} \) is the Heaviside step function\(^1\). Differentiating once with respect to \( K \), we get

\[ \frac{\partial C_{T,K|t,x}}{\partial K} = -\int_K^\infty p_{T,y|t,x}dy = -\mathbb{E}[\mathcal{H}(S_T - K)] \Rightarrow \mathbb{E}[\mathcal{H}(S_T - K)S_T] = C_{T,K|t,x} - K \frac{\partial C_{T,K|t,x}}{\partial K} \]  \hspace{1cm} (35)

Differentiating again with respect to \( K \), we have

\[ \frac{\partial^2 C_{T,K|t,x}}{\partial K^2} = p_{T,K|t,x} = \mathbb{E}[\delta(S_T - K)] \]  \hspace{1cm} (36)

where \( \delta \) is the Dirac delta function\(^1\). Applying Ito’s Lemma to the terminal payoff of the option gives the identity

\(^1\) Heaviside step function: \( \mathcal{H}(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases} \)
\[ d(S_T - K)^+ = \mathcal{A}(S_T - K)dS_T + \frac{1}{2}\delta(S_T - K)\sigma_T^2S_T^2dT \]

\[ = \mathcal{A}(S_T - K)\mu_T S_T dT + \frac{1}{2}\delta(S_T - K)\sigma_T^2S_T^2dT + \mathcal{A}(S_T - K)\sigma_T S_T d\tilde{W}_T \]

Taking conditional expectations on both sides gives

\[ dC_{T,K|t,x} = d\mathbb{E}[(S_T - K)^+] = \mu_T \mathbb{E}[\mathcal{A}(S_T - K)S_T]dT + \frac{1}{2}\mathbb{E}[\delta(S_T - K)\sigma_T^2S_T^2]dT \]

\[ \Rightarrow \frac{\partial C_{T,K|t,x}}{\partial T} = \mu_T \mathbb{E}[\mathcal{A}(S_T - K)S_T] + \frac{1}{2}\mathbb{E}[\delta(S_T - K)\sigma_T^2S_T^2] \] \hspace{1cm} (38)

Notice that

\[ \mathbb{E}[\delta(S_T - K)\sigma_T^2S_T^2] = K^2 \mathbb{E}[\delta(S_T - K)\sigma_T^2] \] \hspace{1cm} (39)

and from Bayes’ rule

\[ \mathbb{E}[\delta(S_T - K)\sigma_T^2] = \mathbb{E}[\sigma_T^2|S_T = K] \mathbb{E}[\delta(S_T - K)] \] \hspace{1cm} (40)

we can derive from (38)

\[ \frac{\partial C_{T,K|t,x}}{\partial T} = \mu_T C_{T,K|t,x} - \mu_T K \frac{\partial C_{T,K|t,x}}{\partial K} + \frac{1}{2}K^2 \frac{\partial^2 C_{T,K|t,x}}{\partial K^2} \mathbb{E}[\sigma_T^2|S_T = K] \]

\[ \Rightarrow \mathbb{E}[\sigma_T^2|S_T = K] = \frac{\partial C_{T,K|t,x}}{\partial T} + \mu_T K \frac{\partial C_{T,K|t,x}}{\partial K} - \mu_T C_{T,K|t,x} \]

\[ \frac{\partial C_{T,K|t,x}}{\partial T} + K^2 \frac{\partial^2 C_{T,K|t,x}}{\partial K^2} \]

\[ \Rightarrow \mathbb{E}[\sigma_T^2|S_T = K] = \frac{\partial C_{T,K|t,x}}{\partial T} + \mu_T K \frac{\partial C_{T,K|t,x}}{\partial K} - \mu_T C_{T,K|t,x} \]

\[ = \frac{\partial^2 C_{T,K|t,x}}{\partial K^2} \]

This is identical to (33). It means that the conditional expectation of the stochastic variance must equal the Dupire local variance [2]. That is, local variance is the risk-neutral expectation of the instantaneous variance conditional on the final stock price \( S_T \) being equal to the strike price \( K \) [3].

2.2.2. Formula in Log-moneyness

In real applications, numerical methods are often in favor of log-moneyness to be the spatial variable, which can be regarded as a centered log-strike

1 Dirac delta function can be viewed as the derivative of the Heaviside step function: \( \delta(x) = \frac{d\mathcal{H}(x)}{dx} = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases} \)

which is also constrained to satisfy the identity: \( \int_{-\infty}^{\infty} \delta(x)dx = 1 \).
\[ k = \ln \frac{K}{F_{t,T}} \quad \text{where} \quad \frac{\partial k}{\partial K} = \frac{1}{K}, \quad \frac{\partial k}{\partial T} = -\mu_T \] (42)

We want to express the Dupire formula in the \((T,k)\)-plane using call option price \(C_{T,k}\) (short for \(C_{T,k|t,z}\) for \(z = \ln \frac{x}{F_{t,T}}\)) equivalent to the forward call \(C_{T,K}\) (short for \(C_{T,K|t,x}\)). Note that although the \(C_{T,k}\) and \(C_{T,K}\) are equivalent, they are two different functions. The conversion from \((T,K)\)-plane to \((T,k)\)-plane is achieved by using the following partial derivatives derived by chain rule

\[
\begin{align*}
\frac{\partial C_{T,K}}{\partial T} &= \frac{\partial C_{T,k}}{\partial T} + \frac{\partial C_{T,K}}{\partial k} \frac{\partial k}{\partial T} = \frac{\partial C_{T,k}}{\partial T} - \mu_T \frac{\partial C_{T,k}}{\partial k} \\
\frac{\partial C_{T,K}}{\partial K} &= \frac{\partial C_{T,k}}{\partial T} \frac{\partial T}{\partial K} + \frac{\partial C_{T,K}}{\partial k} \frac{\partial k}{\partial T} = \frac{1}{K} \frac{\partial C_{T,k}}{\partial k} \\
\frac{\partial^2 C_{T,K}}{\partial K^2} &= \frac{\partial}{\partial T} \left( \frac{\partial C_{T,k}}{\partial k} \right) \frac{\partial T}{\partial K} + \frac{\partial}{\partial k} \left( \frac{\partial C_{T,K}}{\partial k} \right) \frac{\partial k}{\partial K} = \frac{1}{K} \frac{\partial k}{\partial K} \left( \frac{1}{K} \frac{\partial C_{T,k}}{\partial k} \right) = \frac{1}{K^2} \left( \frac{\partial^2 C_{T,k}}{\partial k^2} - \frac{\partial C_{T,k}}{\partial k} \right)
\end{align*}
\] (43)

Plugging these partial derivatives into (33), we have the Dupire formula expressed in \(k\)

\[
\frac{\sigma_{T,k}^2}{2} = \frac{\partial C_{T,k}}{\partial T} - \mu_T C_{T,k} - \frac{\partial^2 C_{T,k}}{\partial K^2} \frac{\partial C_{T,K}}{\partial k} \] (44)

where \(\sigma_{T,k}\) is the local volatility in \((T,k)\)-plane equivalent to \(\sigma_{T,K}\).

2.3. Local Volatility by Implied Volatility

It is a market standard to quote option prices as Black-Scholes implied volatilities. Hence, it is more straightforward to express the local volatility in terms of the implied volatilities rather than option prices. Taking \(t\) as of today, we can define the forward price as

\[ F_{t,T} = S_t \exp \left( \int_t^T \mu_u du \right) \] (45)

The forward call price in Black-Scholes model is then given by

\[ X_{T,K,x} = F_{t,T} \Phi(d_+) - K \Phi(d_-) \quad \text{with} \quad d_\pm = \frac{\ln \frac{F_{t,T}}{K} + \xi_{T,K} \sqrt{\tau}}{\xi_{T,K} \sqrt{\tau}} \pm \frac{\xi_{T,K} \sqrt{\tau}}{2}, \quad \tau = T - t \] (46)
where $\xi_{T,K}$ is the Black-Scholes implied volatility derived from market quotes of vanilla options and $\Phi$ the standard normal cumulative density function. Its partial derivatives can be derived as

$$\frac{\partial X_{T,K,\xi}}{\partial T} = \mu_T F_{t,T} \Phi(d_+) + F_{t,T} \phi(d_+) \frac{\partial d_+}{\partial T} - K \Phi(d_-) \frac{\partial d_-}{\partial T} = \mu_T F_{t,T} \Phi(d_+) + \frac{K \phi(d_-) \xi}{2\sqrt{\tau}}$$

$$\frac{\partial X_{T,K,\xi}}{\partial K} = F_{t,T} \phi(d_+) \frac{\partial d_+}{\partial K} - K \phi(d_-) \frac{\partial d_-}{\partial K} - \Phi(d_-) = -\Phi(d_-), \quad \frac{\partial^2 X_{T,K,\xi}}{\partial K^2} = \frac{\phi(d_-) d_+}{\xi}$$

$$\frac{\partial X_{T,K,\xi}}{\partial \xi} = F_{t,T} \phi(d_+) \frac{\partial d_+}{\partial \xi} - K \phi(d_-) \frac{\partial d_-}{\partial \xi} = K \phi(d_-) \sqrt{\tau}, \quad \frac{\partial^2 X_{T,K,\xi}}{\partial \xi^2} = \frac{d_+ d_- K \phi(d_-) \sqrt{\tau}}{\xi}$$

$$\frac{\partial^2 X_{T,K,\xi}}{\partial \xi \partial K} = \frac{\phi(d_-) d_+}{\xi}$$

where we have used

$$\frac{\partial d_\pm}{\partial T} = \frac{\mu_T d_\pm}{\xi \sqrt{\tau}} - \frac{d_\pm}{2\tau}, \quad \frac{\partial d_\pm}{\partial K} = -\frac{1}{K \xi \sqrt{\tau}}, \quad \frac{\partial d_\pm}{\partial \xi} = -\frac{d_\pm}{\xi}, \quad \frac{\partial^2 d_\pm}{\partial K \partial \xi} = \frac{1}{K \xi^2 \sqrt{\tau}}$$

and the identity

$$F_{t,T} \phi(d_+) = K \phi(d_-)$$

Further using the partial derivatives

$$\frac{\partial C_{T,K}}{\partial T} = \frac{\partial X_{T,K,\xi}}{\partial T} + \frac{\partial X_{T,K,\xi}}{\partial \xi} \frac{\partial \xi}{\partial T}, \quad \frac{\partial C_{T,K}}{\partial K} = \frac{\partial X_{T,K,\xi}}{\partial K} + \frac{\partial X_{T,K,\xi}}{\partial \xi} \frac{\partial \xi}{\partial K}$$

$$\frac{\partial^2 C_{T,K}}{\partial K^2} = \frac{\partial^2 X_{T,K,\xi}}{\partial K^2} + 2 \frac{\partial^2 X_{T,K,\xi}}{\partial K \partial \xi} \frac{\partial \xi}{\partial K} + \frac{\partial X_{T,K,\xi}}{\partial \xi} \frac{\partial^2 \xi}{\partial \xi^2} + \frac{\partial^2 X_{T,K,\xi}}{\partial \xi^2} \left( \frac{\partial \xi}{\partial K} \right)^2$$

the local volatility given by implied volatility can be derived from (33) as
\[ \sigma^2_{T,K} = \frac{\frac{\partial C_{T,K}}{\partial T} + \mu_T K \frac{\partial C_{T,K}}{\partial K} - \mu_T C_{T,K}}{K^2 \frac{\partial^2 C_{T,K}}{\partial K^2}} = \frac{\frac{\partial X}{\partial T} + \frac{\partial X}{\partial \xi} \left( \frac{\partial \xi_{T,K}}{\partial T} + \mu_T K \frac{\partial \xi_{T,K}}{\partial K} \right) + \mu_T K \frac{\partial X}{\partial K} - \mu_T X}{K^2 \frac{1}{2} \left( \frac{\partial^2 X}{\partial K^2} + 2 \frac{\partial^2 X}{\partial K \partial \xi} \frac{\partial \xi_{T,K}}{\partial K} + \frac{\partial X}{\partial \xi} \frac{\partial^2 \xi_{T,K}}{\partial K^2} + \frac{\partial^2 X}{\partial \xi^2} \left( \frac{\partial \xi}{\partial K} \right)^2 \right)} \]

\[ = \mu_T F_{t,T} \Phi(d_+) + \frac{K \Phi(d_-) \xi}{2 \sqrt{T}} + \frac{\partial X}{\partial \xi} \left( \frac{\partial \xi_{T,K}}{\partial T} + \mu_T K \frac{\partial \xi_{T,K}}{\partial K} \right) - \mu_T K \Phi(d_-) - \mu_T X \]

\[ = \frac{\xi^2 + 2 \xi \tau \frac{\partial \xi_{T,K}}{\partial T} + \mu_T K \frac{\partial \xi_{T,K}}{\partial K}}{1 + 2 \sqrt{T} \phi_{d_+} \frac{\partial \xi}{\partial K} + d_+ d_- \tau K^2 \left( \frac{\partial \xi_{T,K}}{\partial K} \right)^2 + \xi \tau K^2 \frac{\partial^2 \xi_{T,K}}{\partial K^2}} \]

Numerical methods, e.g. PDE or Monte Carlo simulation, often demand a local volatility function constructed on a 2D grid to perform pricing. In these methods, it is often more numerically stable and convenient to work with a spatial dimension in log-strike or in log-moneyness.

2.3.1. Formula in Log-strike

The local volatility formula in log strike \( x = \ln K \) can be derived from (51)

\[ \sigma^2_{x,T} = \frac{\xi^2 + 2 \xi \tau \frac{\partial \xi_{T,K}}{\partial T} + \mu_T K \frac{\partial \xi_{T,K}}{\partial K}}{1 + 2 \sqrt{T} \phi_{d_+} \frac{\partial \xi}{\partial K} + d_+ d_- \tau K^2 \left( \frac{\partial \xi_{T,K}}{\partial K} \right)^2 + \xi \tau K^2 \frac{\partial^2 \xi_{T,K}}{\partial K^2}} \]

\[ = \frac{\xi^2 + 2 \xi \tau \frac{\partial \xi_{T,K}}{\partial T} + \mu_T K \frac{\partial \xi_{T,K}}{\partial K}}{1 + \left( \xi \tau - 2 \frac{k}{\xi} \right) \frac{\partial \xi}{\partial x} + \left( \xi^2 - \frac{\xi^2 \tau^2}{4} \right) \left( \frac{\partial \xi}{\partial x} \right)^2 + \xi \tau \frac{\partial^2 \xi}{\partial x^2}} \]

\[ = \frac{\xi^2 + 2 \xi \tau \frac{\partial \xi_{T,K}}{\partial T} + \mu_T K \frac{\partial \xi_{T,K}}{\partial K}}{\left( 1 - \frac{k}{\xi} \frac{\partial \xi}{\partial x} \right)^2 - \left( \frac{\xi \tau \frac{\partial \xi}{\partial x}}{2} \right)^2 + \xi \tau \frac{\partial^2 \xi}{\partial x^2}} \]

providing that we have the following identities
\[
\frac{\partial \xi}{\partial K} = \frac{\partial \xi}{\partial x} \frac{\partial x}{\partial K} = \frac{1}{K} \frac{\partial \xi}{\partial x}, \quad \frac{\partial^2 \xi}{\partial K^2} = \frac{\partial}{\partial x} \left( \frac{1}{K} \frac{\partial \xi}{\partial x} \right) \frac{\partial x}{\partial K} = \frac{1}{K^2} \left( \frac{\partial^2 \xi}{\partial x^2} - \frac{\partial \xi}{\partial x} \right), \quad \frac{\partial x}{\partial K} = \frac{1}{K}
\]

\[
d_\pm = \frac{-k}{\xi \sqrt{T}} \pm \frac{\xi \sqrt{T}}{2}, \quad k = \ln \left( \frac{K}{F_{t,T}} \right)
\]

2.3.2. Formula in Log-moneyness

We may also want to change the spatial variable to log-moneyness \( k \). Defining a new quantity, implied total variance \( v_{T,k} \), which is equivalent to \( \xi^2_{T,k}T \), the Black-Scholes call price that is equivalent to (46) then transforms into

\[
X_{T,k,v} = F_{t,T} \left( \Phi(d_+) - \exp(k) \Phi(d_-) \right) \quad \text{with} \quad d_\pm = \frac{-k}{\sqrt{v_{T,k}}} \pm \frac{\sqrt{v_{T,k}}}{2}
\]

The partial derivatives of \( X_{T,k,v} \) can be derived as

\[
\frac{\partial X_{T,k,v}}{\partial v} = F_{t,T} \left( \frac{\partial \Phi(d_+)}{\partial d_+} \frac{\partial d_+}{\partial v} - \exp(k) \frac{\partial \Phi(d_-)}{\partial d_-} \frac{\partial d_-}{\partial v} \right) = F_{t,T} \Phi(d_+) \left( \frac{\partial d_+}{\partial v} - \frac{\partial d_-}{\partial v} \right) = \frac{F_{t,T} \Phi(d_+)}{2\sqrt{v}}
\]

\[
\frac{\partial^2 X_{T,k,v}}{\partial v^2} = \frac{\partial X_{T,k,v}}{\partial v} \left( -\frac{1}{2v} - d_+ \frac{\partial d_+}{\partial v} \right) = \frac{\partial X_{T,k,v}}{\partial v} \left( -\frac{1}{2v} - \left( -\frac{k}{\sqrt{v}} + \frac{\sqrt{v}}{2} \right) \left( \frac{k}{2\sqrt{v^3}} + \frac{1}{4v} \right) \right)
\]

\[
\frac{\partial X_{T,k,v}}{\partial v} = \frac{\partial X_{T,k,v}}{\partial v} \left( \frac{k^2}{2v^2} - \frac{1}{2v} - \frac{1}{8} \right)
\]

\[
\frac{\partial X_{T,k,v}}{\partial k} = F_{t,T} \left( \Phi(d_+) \frac{\partial d_+}{\partial k} - \exp(k) \Phi(d_-) - \exp(k) \Phi(d_-) \frac{\partial d_-}{\partial k} \right) = -F_{t,T} \exp(k) \Phi(d_-)
\]

\[
\frac{\partial^2 X_{T,k,v}}{\partial k^2} = -F_{t,T} \exp(k) \Phi(d_-) + F_{t,T} \frac{\exp(k) \Phi(d_-)}{\sqrt{v}} = \frac{\partial X_{T,k,v}}{\partial k} + 2 \frac{\partial X_{T,k,v}}{\partial v}
\]

\[
\frac{\partial^2 X_{T,k,v}}{\partial k \partial v} = \frac{\partial}{\partial k} \left( \frac{F_{t,T} \Phi(d_+)}{2\sqrt{v}} \right) = \frac{\partial X_{T,k,v}}{\partial v} d_+ \frac{\partial d_+}{\partial k} = \frac{\partial X_{T,k,v}}{\partial v} \left( \frac{1}{2} - \frac{k}{v} \right)
\]

\[
\frac{\partial X_{T,k,v}}{\partial T} = \left( \Phi(d_+) - \exp(k) \Phi(d_-) \right) \frac{\partial F_{t,T}}{\partial T} = \mu_T X_{T,k,v}
\]

We may connect the local volatility \( \sigma_{T,k} \) to the implied total variance \( v_{T,k} \) via two steps. Firstly we bridge the \( \sigma_{T,k} \) to \( X_{T,k,v} \) by (44) using the chain rule
\[
\frac{\partial C_{T,k}}{\partial T} = \frac{\partial X_{T,k,v}}{\partial T} + \frac{\partial X_{T,k,v}}{\partial v} \frac{\partial v}{\partial T} \quad \frac{\partial C_{T,k}}{\partial k} = \frac{\partial X_{T,k,v}}{\partial k} + \frac{\partial X_{T,k,v}}{\partial v} \frac{\partial v}{\partial k}
\]

\[
\frac{\partial^2 C_{T,k}}{\partial k^2} = \frac{\partial}{\partial k} \left( \frac{\partial X_{T,k,v}}{\partial k} + \frac{\partial X_{T,k,v}}{\partial v} \frac{\partial v}{\partial k} \right) + \frac{\partial}{\partial v} \left( \frac{\partial X_{T,k,v}}{\partial k} + \frac{\partial X_{T,k,v}}{\partial v} \frac{\partial v}{\partial k} \right) \frac{\partial v}{\partial k}
\]

\[
= \frac{\partial^2 X_{T,k,v}}{\partial k^2} + \frac{\partial^2 X_{T,k,v}}{\partial k \partial v} \frac{\partial v}{\partial k} + \frac{\partial X_{T,k,v}}{\partial v} \frac{\partial^2 v}{\partial k^2} + \frac{\partial^2 X_{T,k,v}}{\partial v^2} \left( \frac{\partial v}{\partial k} \right)^2 - \frac{\partial^2 X_{T,k,v}}{\partial v} \frac{\partial v}{\partial k} \frac{\partial v}{\partial k}
\]

This gives the local volatility expressed in terms of derivatives of \(X_{T,k,v}\)

\[
\sigma_{T,k}^2 = \frac{2 \left( \frac{\partial X_{T,k,v}}{\partial T} + \frac{\partial X_{T,k,v}}{\partial v} \frac{\partial v}{\partial T} - \mu_T X_{T,k,v} \right)}{\frac{\partial^2 X_{T,k,v}}{\partial k^2} + 2 \frac{\partial^2 X_{T,k,v}}{\partial k \partial v} \frac{\partial v}{\partial k} + \frac{\partial X_{T,k,v}}{\partial v} \frac{\partial^2 v}{\partial k^2} + \frac{\partial^2 X_{T,k,v}}{\partial v^2} \left( \frac{\partial v}{\partial k} \right)^2 - \frac{\partial X_{T,k,v}}{\partial k} - \frac{\partial X_{T,k,v}}{\partial v} \frac{\partial v}{\partial k}}
\]

Secondly we substitute the partial derivatives in (55) into (57) and reach the final equation

\[
\sigma_{T,k}^2 = \frac{2 \left( \mu_T X + \frac{\partial X}{\partial v} \frac{\partial v}{\partial T} - \mu_T X \right)}{\frac{\partial X}{\partial k} + 2 \frac{\partial X}{\partial v} \left( \frac{k^2}{v^2} - \frac{1}{v} \right) \frac{\partial v}{\partial k} + \frac{\partial X}{\partial v} \frac{\partial^2 v}{\partial k^2} + \frac{\partial X}{\partial v} \left( \frac{k^2}{2v^2} - \frac{1}{2v} - \frac{1}{8} \right) \left( \frac{\partial v}{\partial k} \right)^2 - \frac{\partial X}{\partial k} - \frac{\partial X}{\partial v} \frac{\partial v}{\partial k} \frac{\partial v}{\partial k}}
\]

\[
\Rightarrow \sigma_{T,k}^2 = \frac{\frac{\partial v}{\partial T}}{1 - \frac{k \frac{\partial v}{\partial k}}{v} + \frac{1}{4} \left( \frac{k^2}{v^2} - \frac{1}{v} - \frac{1}{4} \right) \left( \frac{\partial v}{\partial k} \right)^2 + \frac{1}{2} \frac{\partial^2 v}{\partial k^2}}
\]

2.3.3. Equivalency in Formulas

The \(\sigma_{T,k}^2\) in (51) is in fact equivalent to the \(\sigma_{T,k}^2\) in (58). This can be shown as follows

\[
\sigma_{T,k}^2 = \frac{\frac{\partial v}{\partial T}}{1 - \frac{k \frac{\partial v}{\partial k}}{v} + \left( \frac{k^2}{4v^2} - \frac{1}{4v} - \frac{1}{16} \right) \left( \frac{\partial v}{\partial k} \right)^2 + \frac{1}{2} \frac{\partial^2 v}{\partial k^2} \frac{\partial^2 v}{\partial k^2}}
\]

\[
= \frac{\xi^2 + 2 \xi \tau \left( \frac{\partial \xi}{\partial T} + \mu K \frac{\partial \xi}{\partial K} \right)}{1 - 2 \xi \tau K \frac{\partial \xi}{\partial K} + \left( \frac{k^2}{v^2} - \frac{1}{v} - \frac{1}{4} \right) \left( \xi \tau K \frac{\partial \xi}{\partial K} \right)^2 + \tau K^2 \left( \frac{\partial \xi}{\partial K} \right)^2 + \xi \frac{\partial \xi}{\partial K} - \xi \frac{\partial^2 \xi}{\partial K^2}}
\]
\[
\frac{\xi^2 + 2\xi\tau (\frac{\partial \xi}{\partial T} + \mu K \frac{\partial \xi}{\partial K})}{1 + \left(1 - \frac{k}{v}\right) \xi \tau K \frac{\partial \xi}{\partial K} + \left(\frac{k^2}{v} - \nu\right) \tau K^2 \left(\frac{\partial \xi}{\partial K}\right)^2 + \xi \tau K^2 \frac{\partial^2 \xi}{\partial K^2}} = \frac{\xi^2 + 2\xi\tau (\frac{\partial \xi}{\partial T} + \mu K \frac{\partial \xi}{\partial K})}{1 + 2\sqrt{\tau K} d_+ \frac{\partial \xi}{\partial K} + d_+ d_- \tau K^2 \left(\frac{\partial \xi}{\partial K}\right)^2 + \xi \tau K^2 \frac{\partial^2 \xi}{\partial K^2}} = \sigma_{T, K}^2
\]

where by definition we have

\[
k = \ln \frac{K}{F_{T,T}}, \quad \nu = \xi^2 \tau, \quad d_\pm = \frac{\ln \frac{F_{T,T} \pm \xi^2 \tau}{K}}{\xi \sqrt{\nu}} = \frac{-k \pm \sqrt{v}}{2}, \quad d_+ d_- = \frac{k^2}{v} - \frac{v}{4}
\]

and also have the identities

\[
\frac{\partial v}{\partial T} = \frac{\partial (\xi^2 \tau)}{\partial T} = \xi^2 + 2\xi\tau \left(\frac{\partial \xi}{\partial T} + \mu K \frac{\partial \xi}{\partial K} \frac{\partial \xi}{\partial T}\right) = \xi^2 + 2\xi\tau \left(\frac{\partial \xi}{\partial T} + \mu K \frac{\partial \xi}{\partial K}\right)
\]

\[
\frac{\partial v}{\partial k} = \frac{\partial (\xi^2 \tau)}{\partial k} = \frac{\partial (\xi^2 \tau)}{\partial T} \frac{\partial T}{\partial k} + \frac{\partial (\xi^2 \tau)}{\partial K} \frac{\partial K}{\partial k} = 2\xi\tau \frac{\partial \xi}{\partial K} \frac{\partial \xi}{\partial k} = 2\xi\tau \frac{\partial \xi}{\partial K}
\]

\[
\frac{\partial^2 v}{\partial k^2} = \frac{\partial (2\xi\tau K \frac{\partial \xi}{\partial K})}{\partial T} \frac{\partial T}{\partial k} + \frac{\partial (2\xi\tau K \frac{\partial \xi}{\partial K})}{\partial K} \frac{\partial K}{\partial k} = 2\xi\tau \frac{\partial \xi}{\partial K} + K \frac{\partial \xi}{\partial K} \frac{\partial \xi}{\partial k} + \xi \frac{\partial^2 \xi}{\partial K^2}
\]

\[
= 2\tau K^2 \left(\frac{\partial \xi}{\partial K}\right)^2 + \xi \frac{\partial \xi}{\partial K} + \xi \frac{\partial^2 \xi}{\partial K^2}
\]

considering the fact that in \((T, k)\)-plane the \(T\) and \(K\) are no longer independent

\[
\frac{\partial K}{\partial T} = \frac{\partial (F_{T,T} \exp(k))}{\partial T} = \mu_T K, \quad \frac{\partial K}{\partial k} = \frac{\partial (F_{T,T} \exp(k))}{\partial k} = K
\]

3. LOCAL VOLATILITY: PDE BY FINITE DIFFERENCE METHOD

In this chapter, we will present a PDE based local volatility model, in which the local volatility surface is constructed as a 2-D function that is piecewise constant in maturity and piecewise linear in log-moneyness (for equity) or delta (for FX). Due to great similarity between FX and equity processes, our interest lies primarily in the context of equity derivatives, the conclusions and formulas drawn from our discussion here are in general applicable to FX products with minor changes. In contrast to the
traditional way to construct the local volatility by estimating highly sensitive and numerically unstable partial derivatives in Dupire formulas, this method relies heavily on solving forward PDE’s to calibrate a parametrized local volatility surface to vanilla option prices in a bootstrapping manner. Once the local volatility surface is calibrated, the backward PDE can then be used to price exotic options (e.g. barrier options) that are in consistent with the market observed implied volatility surface.

Before proceeding to the PDE’s, it is important to have an overview of the date conventions for equity and equity options. The date conventions for FX products are defined in a similar manner.

3.1. Date Conventions of Equity and Equity Option

The diagram illustrates the date definitions for an equity and its associated option. The quantities appeared in the diagram are listed in Table 1.
Table 1. Dates of Equities and Options

<table>
<thead>
<tr>
<th>attribute</th>
<th>symbol</th>
<th>description</th>
<th>remark/example</th>
</tr>
</thead>
<tbody>
<tr>
<td>trade date</td>
<td>$t_0$</td>
<td>on which the equity/option is traded</td>
<td>today</td>
</tr>
<tr>
<td>equity spot lag</td>
<td>$\Delta_{e,s}$</td>
<td>equity premium settlement lag</td>
<td>3D</td>
</tr>
<tr>
<td>equity spot date</td>
<td>$t_{e,s}$</td>
<td>on which the equity premium is settled</td>
<td>$t_{e,s} = t_0 \oplus \Delta_{e,s}$</td>
</tr>
<tr>
<td>equity maturity date</td>
<td>$t_{e,m}$</td>
<td>equity maturity date</td>
<td>$t_{e,m} = t_0 \oplus 1Y$</td>
</tr>
<tr>
<td>equity pay lag$^1$</td>
<td>$\Delta_{e,p}$</td>
<td>lag between $t_{e,m}$ and $t_{e,p}$</td>
<td>e.g. same as $\Delta_{e,s}$</td>
</tr>
<tr>
<td>equity pay date</td>
<td>$t_{e,p}$</td>
<td>on which the equity payoff is settled</td>
<td>$t_{e,p} = t_{e,m} \oplus \Delta_{e,p}$</td>
</tr>
<tr>
<td>$i$-th dividend</td>
<td>$\theta_i$</td>
<td>dividend payment amount</td>
<td></td>
</tr>
<tr>
<td>$i$-th ex-div. date</td>
<td>$t_{i,e}$</td>
<td>ex-dividend date</td>
<td></td>
</tr>
<tr>
<td>$i$-th div. pay date</td>
<td>$t_{i,p}$</td>
<td>dividend pay date</td>
<td></td>
</tr>
<tr>
<td>option spot lag</td>
<td>$\Delta_{o,s}$</td>
<td>option premium settlement lag</td>
<td>2D</td>
</tr>
<tr>
<td>option spot date</td>
<td>$t_{o,s}$</td>
<td>on which the option is settled</td>
<td>$t_{o,s} = t_0 \oplus \Delta_{o,s}$</td>
</tr>
<tr>
<td>option maturity date</td>
<td>$t_{o,m}$</td>
<td>option maturity date</td>
<td>$t_{o,m} = t_0 \oplus 1Y$</td>
</tr>
<tr>
<td>option pay lag</td>
<td>$\Delta_{o,p}$</td>
<td>lag between $t_{o,m}$ and $t_{o,p}$</td>
<td>e.g. same as $\Delta_{o,s}$</td>
</tr>
<tr>
<td>option pay date</td>
<td>$t_{o,p}$</td>
<td>on which the equity payoff is settled</td>
<td>$t_{o,p} = t_{o,m} \oplus \Delta_{o,p}$</td>
</tr>
<tr>
<td>day roll convention</td>
<td>$\oplus$</td>
<td>rolling with convention “following” defining business days and holidays</td>
<td>Following US / UK / HK</td>
</tr>
<tr>
<td>calendar</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

As most of the quantities are self-explanatory, our discussion focuses more on the treatment of equity dividends.

3.2. Deterministic Dividends

In our example, we can assume both the short rate and the dividend rate are deterministic and continuous, e.g. time-dependent $r_u$ and $q_u$ as in (22). The equity forward in this case can be calculated by

$$F(t_0, t_{e,m}) = S(t_0) \frac{P_q(t_{e,s}, t_{e,p})}{P_r(t_{e,s}, t_{e,p})} \quad \text{where}$$

$$P_q(t, T) = \exp \left( -\int_t^T q_u du \right), \quad P_r(t, T) = \exp \left( -\int_t^T r_u du \right)$$

(63)

In a more realistic implementation, we may assume the underlying equity issues a series of discrete dividends with fixed amounts in a foreseeable future. It is obvious that the equity spot still follows the SDE (22) with $q_u = 0$ in between two adjacent ex-dividend dates (There is discontinuity in

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$^1$ Equity settlement delay
spot process on ex-dividend dates that demands special treatment. This will be discussed in detail in due course). With fixed dividends, the equity forward becomes

\[ F(t_0, t_{e,m}) = \frac{S(t_0) - \sum_i \theta_i P_r(t_{e,s}, t_{i,p})}{P_r(t_{e,s}, t_{e,p})} \quad \text{for} \quad t_0 < t_{i,e} \leq t_{e,m} \quad (64) \]

where \( \theta_i \) is the fixed amount of the \( i \)-th dividend issued on ex-dividend date \( t_{i,e} \).

Discrete dividend can also be modeled as proportional dividend. It assumes that at each ex-dividend date, the dividend payment will result in a price drop in equity spot proportional to the spot level. For example, the equity spot before and after the dividend fall has the relationship

\[ S(t_{i,e} + \Delta) = S(t_{i,e} - \Delta)(1 - \eta_i) \quad (65) \]

where \( \Delta \) denotes an infinitesimal amount of time and \( \eta_i \) the proportional dividend rate at ex-dividend date \( t_{i,e} \). By this relationship, we can write the equity forward as

\[ F(t_0, t_{e,m}) = S(t_0) \frac{\prod_i (1 - \eta_i)}{P_r(t_{e,s}, t_{e,p})} \quad \text{for} \quad t_0 < t_{i,e} \leq t_{e,m} \quad (66) \]

Sometimes it is often more convenient to approximate the fixed dividends by proportional dividends. The conversion can be achieved by equating the equity forward in (64) and in (66), such that

\[ \prod_i (1 - \eta_i) = 1 - \frac{1}{S(t_0)} \sum_i \theta_i P_r(t_{e,s}, t_{i,p}) \quad \text{for} \quad t_0 < t_{i,e} \leq t_{e,m} \quad (67) \]

The proportional dividend \( \eta_i \) can then be bootstrapped from a series of fixed dividends \( \theta_i \) starting from the first ex-dividend date.

3.3. Forward PDE

In the following, our derivation is based on the spot process \( S_t \) defined in (22) and its variants.

For example, depending on the application we may write the SDE (22) in terms of log-spot \( z_u = \ln S_u \) or in terms of centered log-spot \( z_u = \ln \frac{S_u}{F_t,u} \)

\[ dz_u = \left( \mu_u - \frac{1}{2} \sigma(u, z)^2 \right) du + \sigma(u, z) d\tilde{W}_u \quad \text{and} \quad dz_u = -\frac{1}{2} \sigma(u, z)^2 du + \sigma(u, z) d\tilde{W}_u \quad (68) \]
where \( \sigma(u, z) \) and \( \sigma(u, z) \) are the local volatility function of \( z \) and \( z \), respectively.

Let’s denote the forward temporal variable by \( u \) for \( t < u < T \), the spatial variable by log-moneyness \( k = \ln \frac{K}{F_{t,u}} \) (as in (42)) and the spot by \( z = \ln \frac{S_u}{F_{t,u}} \). Given that \( z_t = 0 \), the value of a normalized forward call can be defined as

\[
V_{u,k|t,z} = \frac{C_{u,k|t,z}}{F_{t,u}} = \frac{\mathbb{E}[(S_u - K)^+]}{F_{t,u}} = (S_{t} - K)^+ \tag{69}
\]

Let \( \sigma_{u,k} \) be the local volatility function of variable \( k \) equivalent to \( \sigma_{T,K} \). We can derive the forward PDE for \( V_{u,k|t,z} \) from (44)

\[
\frac{\sigma_{u,k}^2}{2} = F_{t,u} \frac{\partial V_{u,k|t,z}}{\partial u} + \mu_u F_{t,u} V_{u,k|t,z} - \mu_u C_{u,k|t,z} = \frac{\partial V_{u,k|t,z}}{\partial u} \frac{\partial^2 V_{u,k|t,z}}{\partial k^2} - F_{t,u} \frac{\partial V_{u,k|t,z}}{\partial k} \frac{\partial V_{u,k|t,z}}{\partial k} \tag{70}
\]

with initial condition

\[
V_{t,k|t,z} = \frac{C_{t,k|t,z}}{F_{t,t}} = \mathbb{E} \left[ (S_t - F_{t,t} e^k)^+ \right] = (1 - e^k)^+ \tag{71}
\]

using the partial derivatives

\[
\frac{\partial V_{u,k|t,z}}{\partial u} = \frac{1}{F_{t,u}} \frac{\partial C_{u,k|t,z}}{\partial u} - \mu_u V_{u,k|t,z}, \quad \frac{\partial V_{u,k|t,z}}{\partial k} = \frac{1}{F_{t,u}} \frac{\partial C_{u,k|t,z}}{\partial k}, \quad \frac{\partial^2 V_{u,k|t,z}}{\partial k^2} = \frac{1}{F_{t,u}} \frac{\partial^2 C_{u,k|t,z}}{\partial k^2} \tag{72}
\]

The PDE (70) appears drift-less and provides more robust calibration stability at low volatility and/or high drift due to the “transparency” of drift in the PDE.

3.3.1. Treatment of Deterministic Dividends

A (discrete) dividend pay-out will typically result in a drop in equity price on the ex-dividend date. Suppose that time \( u \) is the ex-dividend date, the no-arbitrage condition states that at \( u_+ \) the time right after the ex-dividend date (e.g. the difference between \( u \) and \( u_+ \) can be infinitesimal), we must have
\[ S_{u^+} = S_u - \theta_u \]  

where \( \theta_u \) is the value of dividend issued at \( u \) (note that in a rigorous setup the value must take into account the discounting effect due to dividend payment delay). Since a forward is expectation of spot under risk neutral measure\(^1\), we may write

\[ F_{t,u^+} = \mathbb{E}_t[S_{u^+}] = \mathbb{E}_t[S_u - \theta_u] = F_{t,u} - \mathbb{E}_t[\theta_u] \]  

Under the assumption that \( \theta_u \) is a fixed amount, it reads

\[ F_{t,u^+} = F_{t,u} - \theta_u \]  

In our finite difference method, the spatial grid for log-moneyness \( k \) is assumed uniform such that \( k_i - k_{i-1} \) is constant for all \( i \). Dividend payment causes discontinuity in the underlying spot. Evolving the forward PDE (70) from initial time \( t \) produces a state vector \( V_{u,k|t,z} \) at time \( u \). Immediately after the issuance of dividend at time \( u^+ \), the spot and forward drop the same \( \theta_u \) amount and hence the state vector \( V_{u^+,k|t,z} \) must be realigned to reflect the dividend fall. This can be done using the option no-arbitrage condition, such that

\[ C_{u^+,k|t,z} = \mathbb{E}_t\left( (S_{u^+} - K)^+ \right) = \mathbb{E}_t\left( (S_u - \theta_u - F_{t,u^+}e^k)^+ \right) = \mathbb{E}_t\left( (S_u - F_{t,u}e^\hat{k})^+ \right) = C_{u,k|t,z} \]  

where \( \hat{k} = \ln \frac{F_{t,u^+}e^k + \theta_u}{F_{t,u}} \)

Subsequently we can use \( \hat{k} \) to interpolate from the \( V_{u,k|t,z} \) state vector and transform the interpolated value to form \( V_{u^+,k|t,z} \) vector by

\[ V_{u^+,k|t,z} = \frac{C_{u^+,k|t,z}}{F_{t,u^+}} = \frac{C_{u,k|t,z}}{F_{t,u}} \frac{F_{t,u}}{F_{t,u^+}} = \frac{F_{t,u}}{F_{t,u^+}} V_{u,k|t,z} \]  

If the dividend is proportional, we must have spot price \( S_{u^+} = S_u (1 - \eta_u) \) for a rate \( \eta_u \) and hence forward price \( F_{t,u^+} = F_{t,u} (1 - \eta_u) \) before and after the dividend fall. Because we can show that

\(^1\) Strictly speaking, a forward on time \( T \) spot is an expectation of the spot under \( T \)-forward measure, i.e. \( F_{t,T} = \mathbb{E}_t^T[S_T] \). However since the interest rate is assumed deterministic, the \( T \)-forward measure coincides with the risk neutral measure.
V_{u+,k|t,z} = \frac{\mathbb{E}_t \left[(S_{u+} - K)^+\right]}{F_{t,u+}} = \frac{(1 - \eta_u)\mathbb{E}_t \left[(S_u - F_{t,u}e^k)^+\right]}{F_{t,u}(1 - \eta_u)} = \frac{\mathbb{E}_t \left[(S_u - F_{t,u}e^k)^+\right]}{F_{t,u}} = V_{u,k|t,z} \quad (78)

the state vector remains unchanged before and after the issuance of dividend.

With continuous dividend $q_u$, the realignment of state vector is unnecessary because there is no discontinuity in equity spot.

3.4. Backward PDE

Again we assume the spot follows the SDE (22). Without loss of generality, let’s denote $G(S_T|K)$ an arbitrary payoff function with parameter $K$, whose value is contingent on $S_T$ at time $T$. One example of such function would be the payoff function of a call option: $G(S_T|K) = (S_T - K)^+$. Let $U_{u,x|T,K}$ be the expectation of the function $G(S_T|K)$ at time $u$ with spatial variable $x = S_u$, which can be written as

$$U_{u,x|T,K} = \mathbb{E}[G(S_T|K)|u,x] = \int_{\mathbb{R}} G(y|K)p_{T,y|u,x}dy \quad (79)$$

where the transition probability $p_{T,y|u,x}$ follows the Kolmogorov backward equation (21)

$$\frac{\partial p_{T,y|u,x}}{\partial u} = \mu_u x \frac{\partial p_{T,y|u,x}}{\partial x} + \frac{\sigma^2_{u,x}x^2}{2} \frac{\partial^2 p_{T,y|u,x}}{\partial x^2} \quad (80)$$

In turn, we can derive the backward PDE for the $U_{u,x|T,K}$ such that

$$\frac{\partial U_{u,x|T,K}}{\partial u} = \int_{\mathbb{R}} G(y|K) \frac{\partial p_{T,y|u,x}}{\partial u}dy = -\int_{\mathbb{R}} G(y|K) \left( \mu_u x \frac{\partial p_{T,y|u,x}}{\partial x} + \frac{\sigma^2_{u,x}x^2}{2} \frac{\partial^2 p_{T,y|u,x}}{\partial x^2} \right)dy \quad (81)$$

with terminal condition

$$U_{T,x|T,K} = G(x|K) \quad (82)$$

3.4.1. PDE in Centered Log-spot

Assuming the spatial variable is $z_u = \ln \frac{x}{F_{t,u}}$ at time $u$, we may write $U_{u,x|T,K}$ in the $(u, z)$-plane equivalent to $U_{u,x|T,K}$. The backward PDE (81) can then be transformed into
\[
\frac{\partial U_{u,z|T,k}}{\partial u} = -\frac{\sigma^2_{u,z}}{2} \left( \frac{\partial^2 U_{u,z|T,k}}{\partial z^2} - \frac{\partial U_{u,z|T,k}}{\partial z} \right)
\]

(83)

with terminal condition
\[
U_{T,z|T,k} = G(F_{t,T}e^{\hat{z}}|F_{t,T}e^{k})
\]

(84)

by using the following partial derivatives derived from the chain rule
\[
\frac{\partial z}{\partial x} = \frac{1}{x}, \quad \frac{\partial z}{\partial u} = -\mu_u, \quad \frac{\partial U_{u,z|T,k}}{\partial u} = \frac{\partial U_{u,z|T,k}}{\partial u} + \frac{\partial U_{u,z|T,k}}{\partial z} \frac{\partial z}{\partial u} = \frac{\partial U_{u,z|T,k}}{\partial u} - \mu_u \frac{\partial U_{u,z|T,k}}{\partial z} - \mu_u \frac{\partial U_{u,z|T,k}}{\partial z}
\]

(85)

3.4.1.1. Treatment of Deterministic Dividends

With fixed dividend \( \theta_u \), we have
\[
S_{u+} = S_u - \theta_u \quad \text{and} \quad F_{t,u+} = F_{t,u} - \theta_u
\]

(86)

The no arbitrage condition shows that for the spatial grid \( z \)
\[
U_{u,z|T,k} = \mathbb{E}[G(S_T|F_{t,T}e^k)|u,F_{t,u}e^{\hat{z}}] = \mathbb{E}[G(S_T|F_{t,T}e^k)|u_{+},F_{t,u}e^{\hat{z}} - \theta_u]
\]

\[
= \mathbb{E}[G(S_T|F_{t,T}e^k)|u_{+},F_{t,u+e^{\hat{z}}}] = U_{u+\hat{z}|T,k} \quad \text{where} \quad \hat{z} = \ln \frac{F_{t,u}e^{\hat{z}} - \theta_u}{F_{t,u+}}
\]

(87)

It is likely that if \( z \) is sufficiently small (e.g. at lower boundary of spatial grid) we may end up with \( F_{t,u}e^{\hat{z}} - \theta_u < 0 \), which makes the \( \hat{z} \) not well defined. A solution is to floor it to a small positive number, e.g. taking \( \max(10^{-10}, F_{t,u}e^{\hat{z}} - \theta_u) \). This is valid because equity spot must be positive and the \( U_{u+\hat{z}|T,k} \) flattens as \( \hat{z} \) goes to negative infinity. After the special treatment, we can use the \( \hat{z} \) to interpolate from the \( U_{u,z|T,k} \) state vector and convert the interpolated value into vector \( U_{u+\hat{z}|T,k} \).

With proportional dividend, the conclusion drawn for forward PDE still applies here and the state vector remains unchanged before and after the dividend fall. With continuous dividend, the realignment of state vector is unnecessary because there is no discontinuity in equity spot.
3.4.1.2. Vanilla Call

Due to the duality between the forward and backward PDE, it is evident that vanilla calls (or puts) must admit the identity: \( U_{t,z|\tau,k} = V_{t,k|t,z} F_{t,T} \), where \( U_{t,z|\tau,k} \) is the forward call solved from backward PDE (83) and \( V_{t,k|t,z} \) the normalized forward call solved from forward PDE (70). This relationship can be used to check the correctness of implementation of the numerical engines of forward and backward PDE.

3.4.2. PDE in Log-spot

For pricing some exotic options, e.g. barrier options, it is more convenient to use log-spot \( z = \ln x \) as the spatial variable. Similarly we can define \( k = \ln K \). Let’s denote the (discounted) price of a derivative product by

\[
X_{u,z|\tau,k} = D_{u,T} U_{u,z|\tau,k} = \mathbb{E}[D_{u,T} G(S_T | e^k)|u, e^z] \tag{88}
\]

By taking into account the discount factor, it must follow the following backward PDE

\[
\frac{\partial X_{u,z|\tau,k}}{\partial u} = r_u X_{u,z|\tau,k} + D_{u,T} \frac{\partial U_{u,z|\tau,k}}{\partial u} \nonumber
\]

\[
= r_u X_{u,z|\tau,k} + D_{u,T} \left( -\mu_u x \frac{1}{x} \frac{\partial U_{u,z|\tau,k}}{\partial z} - \frac{\sigma^2_u}{2} x^2 \left( \frac{\partial^2 U_{u,z|\tau,k}}{\partial z^2} - \frac{\partial U_{u,z|\tau,k}}{\partial z} \right) \right) \tag{89}
\]

\[
= -\frac{\sigma^2_u}{2} \frac{\partial^2 X_{u,z|\tau,k}}{\partial z^2} + \left( \frac{\sigma^2_u}{2} - \mu_u \right) \frac{\partial X_{u,z|\tau,k}}{\partial z} + r_u X_{u,z|\tau,k}
\]

where the partial derivatives below have been used

\[
\frac{\partial z}{\partial x} = \frac{1}{x}, \quad \frac{\partial z}{\partial u} = 0,
\]

\[
\frac{\partial U_{u,z|\tau,k}}{\partial u} = \frac{\partial U_{u,z|\tau,k}}{\partial u} + \frac{\partial U_{u,z|\tau,k}}{\partial z} \frac{\partial z}{\partial u} = \frac{\partial U_{u,z|\tau,k}}{\partial u}
\]

\[
\frac{\partial U_{u,z|\tau,k}}{\partial x} = \frac{\partial U_{u,z|\tau,k}}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial U_{u,z|\tau,k}}{\partial z} \frac{\partial z}{\partial x} = \frac{1}{x} \frac{\partial U_{u,z|\tau,k}}{\partial x}
\]

\[
\frac{\partial^2 U_{u,z|\tau,k}}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{1}{x} \frac{\partial U_{u,z|\tau,k}}{\partial z} \right) = -\frac{1}{x^2} \frac{\partial U_{u,z|\tau,k}}{\partial z} + x \frac{\partial U_{u,z|\tau,k}}{\partial x} \frac{\partial z}{\partial u} + \frac{1}{x^2} \frac{\partial^2 U_{u,z|\tau,k}}{\partial z^2} \frac{\partial z}{\partial u}
\]

\[
= \frac{1}{x^2} \left( \frac{\partial^2 U_{u,z|\tau,k}}{\partial z^2} - \frac{\partial U_{u,z|\tau,k}}{\partial z} \right)
\]
3.4.2.1. Treatment of Deterministic Dividends

With fixed dividend $\theta_u$, the no arbitrage condition states that

$$X_{u,z|T,k} = \mathbb{E}\left[D_{u,T} G(S_T|e^K)|u,e^Z\right] = \mathbb{E}\left[D_{u+\varepsilon,T} G(S_T|e^K)|u+\varepsilon,e^Z-\theta_u\right]$$

$$= \mathbb{E}\left[D_{u+\varepsilon,T} G(S_T|e^K)|u+\varepsilon,e^\hat{Z}\right] = X_{u+\varepsilon|T,k} \quad \text{where} \quad \hat{Z} = \ln(e^Z-\theta_u)$$

(91)

Again, extremely small $z$ may result in $\hat{Z}$ that is not well defined, we may floor the difference $e^Z-\theta_u$ to a small positive number, e.g. taking $\max(10^{-10}, e^Z-\theta_u)$. The vector $X_{u,z|T,k}$ can then be interpolated from the known $X_{u+\varepsilon|T,k}$ using the $\hat{Z}$.

With proportional dividend $\eta_u$, again the no arbitrage condition shows

$$X_{u,z|T,k} = \mathbb{E}\left[D_{u,T} G(S_T|e^K)|u,e^Z\right] = \mathbb{E}\left[D_{u+\varepsilon,T} G(S_T|e^K)|u+\varepsilon(1-\eta_u)\right]$$

$$= \mathbb{E}\left[D_{u+\varepsilon,T} G(S_T|e^K)|u+\varepsilon,e^\hat{Z}\right] = X_{u+\varepsilon|T,k} \quad \text{where} \quad \hat{Z} = z + \ln(1-\eta_u)$$

(92)

The vector $X_{u,z|T,k}$ can be interpolated from the $X_{u+\varepsilon|T,k}$ using the $\hat{Z}$.

With continuous dividend, the realignment of state vector is unnecessary because there is no discontinuity in equity spot.

3.5. Local Volatility Surface

This section is devoted to discussing the construction of local volatility surface $\sigma(u,k)$. There are various ways to define the local volatility surface. The one that we would like to discuss is a 2-D function that is piecewise constant in maturity $u$ and piecewise linear in log-moneyness $k = \ln \frac{K}{F_{t,u}}$ (or in delta for FX). The volatility surface comprises a series of volatility smiles $\sigma_j(k)$ for maturity $t < u_1 < \ldots < u_j < \ldots < u_m = T$. At each maturity $u_j$, volatility smile $\sigma_j(k)$ is constructed by linear interpolation between log-moneyness pillars $k_i = \ln \frac{K_i}{F_{t,u}}$ for strikes $K_1 < \ldots < K_i < \ldots < K_n$ and flat extrapolation where the volatility values at $k_1$ and $k_n$ are used for all $k < k_1$ and $k > k_n$, respectively. The smile $\sigma_j(k)$ constructed at $u_j$ is assumed to remain constant over time for any $u$ between the two adjacent maturities $u_{j-1} < u \leq u_j$. 

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Calibration of the local volatility surface is conducted in a bootstrapping manner starting from the shortest maturity \( u_1 \). It is done by solving the forward PDE such that the local volatility surface is able to reproduce the vanilla call prices at the prescribed log-moneyness pillars \( k_i \) for each of the maturities \( u_j \). The PDE can be solved using finite difference method\(^1\) on a uniform grid defined on log-moneyness \( k \) that extends to \( \pm 5 \) standard deviations of the underlying spot. The choice of boundary condition has little impact to the solutions of vanilla option prices because at \( \pm 5 \) standard deviations the transition probability becomes negligibly small. Our application uses linearity boundary condition for its simplicity. To allow a higher tolerance to market data input and smoother calibration process, the objective function may include a penalty term to suppress unfavorable concavity of a local volatility smile. Again, there can be many ways to define the objective function as well as the penalty function. In this essay, we will only focus on the simplest objective (e.g. at maturity \( u_j \)): the least square minimization of vanilla call prices

\[
\arg\min_{\sigma_j(k_i)} \sum_{i=1}^{n} \left( U_{t,z|u_j,k_i}^{BS} - U_{t,z|u_j,k_i}^{PDE} \right)^2
\]

where \( U_{t,z|u_j,k} \) is the normalized forward call price defined in (69), the superscript “BS” denotes the theoretical price by Black-Scholes model and the “PDE” denotes the numerical value by forward PDE. Note that without a penalty term, the minimization can lead to an exact solution given a proper\(^2\) implied volatility surface.

3.6. Barrier Option Pricing

In contrast to the calibration, the pricing of a barrier option relies on the backward PDE (89) in line with proper terminal condition (i.e. payoff function) and boundary conditions defined by the characteristics of the barrier option. Barrier options often demand a spatial grid defined on log-spot \( z = \ln S_u \), which allows an easier fit of time-invariant barrier (e.g. with European or American type of

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\(^1\) A brief introduction to finite difference method can be found in my note “Introduction to Interest Rate Models”, which can be downloaded from website \texttt{http://www.cs.utah.edu/~cxiong/}.

\(^2\) A proper implied volatility surface should well behave and admit no arbitrage.
observation window) into the domain. For example, an up-and-out barrier option would be priced on a domain with upper bound at the barrier level $b$ where Dirichlet boundary condition is applied (the lower bound and its boundary condition remain the same as for vanilla options).
REFERENCES

