Computational Learning Theory: Occam's Razor

Machine Learning
Spring 2018

THE UNIVERSITY OF UTAH

The slides are mainly from Vivek Srikumar

This section

1. Analyze a simple algorithm for learning conjunctions
2. Define the PAC learnability
3. Make formal connections to the principle of Occam's razor

This section

- ✓ Analyze a simple algorithm for learning conjunctions

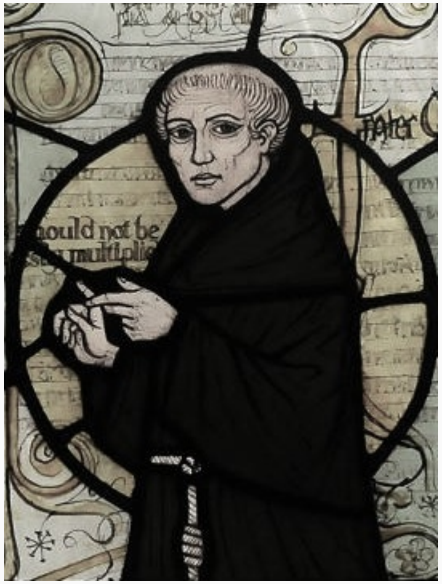
- ✓ Define the PAC model of learning

3. Make formal connections to the principle of Occam's razor

Occam's Razor

Named after William of Occam

– AD 1300s



Prefer simpler explanations over more complex ones

"Numquam ponenda est pluralitas sine necessitate"

(Never posit plurality without necessity.)

Historically, a widely prevalent idea across different schools of philosophy

Towards formalizing Occam's Razor

Claim: The probability that there is a hypothesis $h \in H$ that:

1. Is Consistent with m examples, and
2. Has $err_D(h) > \epsilon$

is less than $|H|\ (1 - \epsilon)^m$

Towards formalizing Occam's Razor

(Assuming consistency)

Claim: The probability that there is a hypothesis $h \in H$ that:

1. Is Consistent with m examples, and
2. Has $err_D(h) > \epsilon$

is less than $|H|\ (1 - \epsilon)^m$

Towards formalizing Occam's Razor

(Assuming consistency)

Claim: The probability that there is a hypothesis $h \in H$ that:

1. Is Consistent with m examples, and
2. Has $err_D(h) > \epsilon$

is less than $|H|(1 - \epsilon)^m$

$\boxed{\text{That is, consistent yet bad}}$

Towards formalizing Occam's Razor

(Assuming consistency)

Claim: The probability that there is a hypothesis $h \in H$ that:

1. Is Consistent with m examples, and
2. Has $err_D(h) > \epsilon$

is less than $|H|\ (1 - \epsilon)^m$

That is, consistent yet bad

Proof: Let h be such a bad hypothesis that has an error $> \epsilon$

Towards formalizing Occam's Razor

(Assuming consistency)

Claim: The probability that there is a hypothesis $h \in H$ that:

1. Is Consistent with m examples, and
2. Has $err_D(h) > \epsilon$

That is, consistent yet bad

is less than $|H|\ (1 - \epsilon)^m$

Proof: Let h be such a bad hypothesis that has an error $> \epsilon$

Probability that h is consistent with one example is $\Pr[f(x) = h(x)] < 1 - \epsilon$

Towards formalizing Occam's Razor

(Assuming consistency)

Claim: The probability that there is a hypothesis $h \in H$ that:

1. Is Consistent with m examples, and
2. Has $err_D(h) > \epsilon$

is less than $|H|\ (1 - \epsilon)^m$

That is, consistent yet bad

Proof: Let h be such a bad hypothesis that has an error $> \epsilon$

Probability that h is consistent with one example is $\Pr[f(x) = h(x)] < 1 - \epsilon$

The training set consists of m examples drawn independently

So, probability that h is consistent with m examples $< (1 - \epsilon)^m$

Towards formalizing Occam's Razor

(Assuming consistency)

Claim: The probability that there is a hypothesis $h \in H$ that:

1. Is Consistent with m examples, and
2. Has $err_D(h) > \epsilon$

is less than $|H|\ (1 - \epsilon)^m$

That is, consistent yet bad

Proof: Let h be such a bad hypothesis that has an error $> \epsilon$

Probability that h is consistent with one example is $\Pr[f(x) = h(x)] < 1 - \epsilon$

The training set consists of m examples drawn independently

So, probability that h is consistent with m examples $< (1 - \epsilon)^m$

Probability that *some bad hypothesis* in H is consistent with m examples is less than $|H|\ (1 - \epsilon)^m$

Occam's Razor

The probability that there is a hypothesis $h \in H$ that is

1. Consistent with m examples, and
2. Has $err_D(h) > \epsilon$

is less than $|H|\,(1 - \epsilon)^m$

Occam's Razor

The probability that there is a hypothesis $h \in H$ that is

1. Consistent with m examples, and
2. Has $err_D(h) > \epsilon$

is less than $|H| (1 - \epsilon)^m$

Just like before, we want to make this probability small, say smaller than δ

Occam's Razor

The probability that there is a hypothesis $h \in H$ that is

1. Consistent with m examples, and
2. Has $err_D(h) > \epsilon$

is less than $|H|\,(1 - \epsilon)^m$

Just like before, we want to make this probability small, say smaller than δ

$$|H|\,(1 - \epsilon)^m < \delta$$

Occam's Razor

The probability that there is a hypothesis $h \in H$ that is

1. Consistent with m examples, and
2. Has $err_D(h) > \epsilon$

is less than $|H|\ (1 - \epsilon)^m$

Just like before, we want to make this probability small, say smaller than δ

$$|H|\ (1 - \epsilon)^m < \delta$$

$$\ln(|H|) + m \ln(1 - \epsilon) < \ln \delta$$

Occam's Razor

The probability that there is a hypothesis $h \in H$ that is

1. Consistent with m examples, and
2. Has $err_D(h) > \epsilon$

is less than $|H|\ (1 - \epsilon)^m$

Just like before, we want to make this probability small, say smaller than δ

$$|H|\ (1 - \epsilon)^m < \delta$$

$$\ln(|H|) + m \ln(1 - \epsilon) < \ln \delta$$

We know that $e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} \cdots > 1 - x$

Let's use $\ln(1 - \epsilon) < -\epsilon$ to get a safer δ

Occam's Razor

The probability that there is a hypothesis $h \in H$ that is

1. Consistent with m examples, and
2. Has $err_D(h) > \epsilon$

is less than $|H|\ (1 - \epsilon)^m$

Just like before, we want to make this probability small, say smaller than δ

$$|H|\ (1 - \epsilon)^m < \delta$$

$$\ln(|H|) + m \ln(1 - \epsilon) < \ln \delta$$

We know that $e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} \cdots > 1 - x$

Let's use $\ln(1 - \epsilon) < -\epsilon$ to get a safer δ

That is, if $m > \frac{1}{\epsilon}\left(\ln(|H|) + \ln \frac{1}{\delta}\right)$ then, the probability of getting a bad hypothesis is small

Occam's Razor

Let H be any hypothesis space.

With probability 1 - δ, a hypothesis $h \in H$ that is consistent with a training set of size m will have an error < ϵ on future examples if

$$m > \frac{1}{\epsilon}\left(\ln(|H|) + \ln\frac{1}{\delta}\right)$$

Occam's Razor

Let H be any hypothesis space.

With probability 1 - δ, a hypothesis $h \in H$ that is consistent with a training set of size m will have an error < ϵ on future examples if

$$m > \boxed{\frac{1}{\epsilon}}\left(\ln(|H|) + \ln\frac{1}{\delta}\right)$$

1. Expecting lower error increases sample complexity (i.e more examples needed for the guarantee)

Occam's Razor

Let H be any hypothesis space.

With probability 1 - δ, a hypothesis $h \in H$ that is consistent with a training set of size m will have an error < ϵ on future examples if

$$m > \boxed{\frac{1}{\epsilon}}\left(\boxed{\ln(|H|)} + \ln\frac{1}{\delta}\right)$$

1. Expecting lower error increases sample complexity (i.e more examples needed for the guarantee)

2. If we have a larger hypothesis space, then we will make learning harder (i.e higher sample complexity)

Occam's Razor

Let H be any hypothesis space.

With probability 1 - δ, a hypothesis $h \in H$ that is consistent with a training set of size m will have an error < ϵ on future examples if

$$m > \boxed{\frac{1}{\epsilon}}\left(\boxed{\ln(|H|)} + \boxed{\ln\frac{1}{\delta}}\right)$$

1. Expecting lower error increases sample complexity (i.e more examples needed for the guarantee)

2. If we have a larger hypothesis space, then we will make learning harder (i.e higher sample complexity)

3. If we want a higher confidence in the classifier we will produce, sample complexity will be higher.

Occam's Razor

Let H be any hypothesis space.

With probability 1 - δ, a hypothesis $h \in H$ that is consistent with a training set of size m will have an error < ϵ on future examples if

$$m > \frac{1}{\epsilon}\left(\ln(|H|) + \ln\frac{1}{\delta}\right)$$

Occam's Razor

Let H be any hypothesis space.

With probability 1 - δ, a hypothesis $h \in H$ that is consistent with a training set of size m will have an error < ϵ on future examples if

$$m > \frac{1}{\epsilon}\left(\ln(|H|) + \ln\frac{1}{\delta}\right)$$

This is called the Occam's Razor because it expresses a preference towards smaller hypothesis spaces.

Occam's Razor

Let H be any hypothesis space.

With probability 1 - δ, a hypothesis $h \in H$ that is consistent with a training set of size m will have an error < ϵ on future examples if

$$m > \frac{1}{\epsilon}\left(\ln(|H|) + \ln\frac{1}{\delta}\right)$$

This is called the Occam's Razor because it expresses a preference towards smaller hypothesis spaces.

Shows when a m-consistent hypothesis generalizes well (i.e. error < ϵ).

Occam's Razor

Let H be any hypothesis space.

With probability 1 - δ, a hypothesis h ∈ H that is consistent with a training set of size m will have an error < ϵ on future examples if

$$m > \frac{1}{\epsilon}\left(\ln(|H|) + \ln\frac{1}{\delta}\right)$$

This is called the Occam's Razor because it expresses a preference towards smaller hypothesis spaces.

Shows when a m-consistent hypothesis generalizes well (i.e. error < ϵ).

Complicated/larger hypothesis spaces are not necessarily bad. But simpler ones are unlikely to fool us by being consistent with many examples!

Consistent Learners and Occam's Razor

From the definition, we get the following general scheme for PAC learning

Given a sample D of m examples

- Find some $h \in H$ that is consistent with ***all*** m examples
  - If m is large enough, a consistent hypothesis must be *close enough* to f
  - Check that m does not have to be too large (i.e polynomial in the relevant parameters): we showed that the "closeness" guarantee requires that

$$m > 1/\epsilon\ (\ln |H| + \ln 1/\delta)$$

- Show that the consistent hypothesis $h \in H$ can be computed efficiently

Exercises

We have seen the decision tree learning algorithm. Suppose our problem has n binary features. What is the size of the hypothesis space?

Are decision trees PAC learnable?