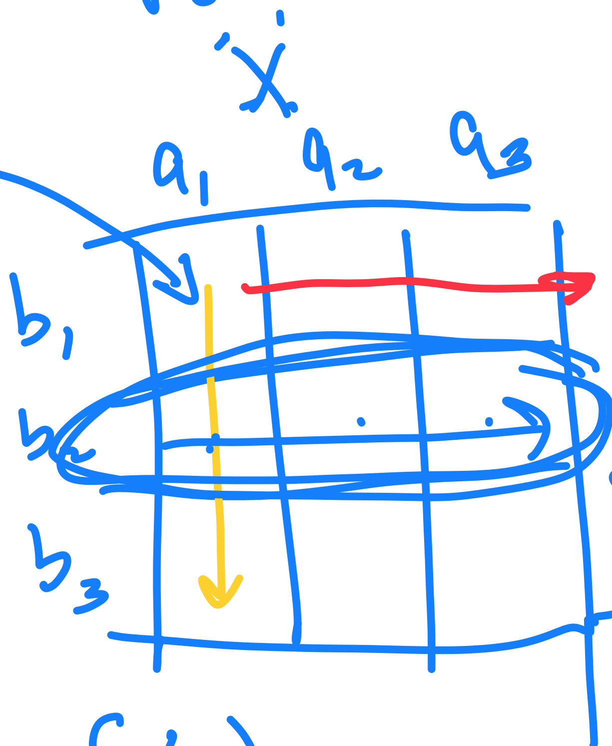
Discrete Joint RV's

$P(\{X=a_i\}\cap\{Y=b_j\})$

$f_{XY}(a,b)$



$f_Y(b)$

$f_Y(b_2)$

$\sum_i P(X=a_i \mid Y=b_2) = 1$

$P(X=a_i \mid Y=b_j) = \dfrac{f_{XY}(a_i,b_j)}{f_Y(b_j)}$

$f_X(a)$

$P(X=a_i)$

Continuous Joint RV's

$f_{XY}(x,y)$

$$1 = \iint f_{XY}(x,y)\,dx\,dy$$

$$\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = f_Y(y).$$

$$P(X=x \mid Y=y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$f(x,y) = \begin{cases} x^2 + \frac{4}{3}xy + y^2 & 0 \le x,y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_X(x) = \int_0^1 \left(x^2 + \frac{4}{3}xy + y^2\right) dy$$

$$f_Y(y) = \int_0^1 \left(x^2 + \frac{4}{3}xy + y^2\right) dx$$

Expectation

$$E(X) = \int_0^1\int_0^1 x\, f_{XY}(x,y)\,dx\,dy$$

$$= \int x \left[\int f_{XY}(x,y)\,dy\right] dx$$

$$= \int_0^1 x\, f_X(x)\,dx$$

Independence

$f_{XY}(x,y)$ $\qquad$ X, Y independent.

$$\frac{f_{XY}(x,y)}{f_Y(y)} = f_X(x) \implies \underbrace{f_{XY}(x,y) = f_X(x) f_Y(y)}_{\forall\, x, y.}$$

Expectation with Joint RVs.

discrete

- $$E[g(X,Y)] = \sum_i \sum_j g(a_i, b_j) f_{XY}(a_i, b_j)$$

cont:

- $$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y)\,dx\,dy.$$

- linearity of E

$$E[rX + sY + t] = rE[X] + sE[Y] + t$$

$$E[X+Y] = E[X] + E[Y]$$

$$X^2 + 2XY + Y^2$$

$$\mathrm{Var}[X+Y] = E[(X+Y)^2] - E[X+Y]^2$$

$$= E[X^2] + 2E[XY] + E[Y^2] - E[X]^2 - 2E[X]E[Y] - E[Y]^2$$

$\mathrm{cov}[X,Y]$

$$= \mathrm{Var}[X] + \mathrm{var}[Y] + 2\left[E[XY] - E[X]E[Y]\right]$$

Covariance

$$\text{cov}[X,Y] = E[XY] - E[X]E[Y]$$

$$= E[(X - E[X])(Y - E[Y])]$$

inches

m

lbs

kg.

σ^2 – variance

σ_{XY} – cov

$\cancel{\Longleftarrow}$

Property: X, Y independent $\Longrightarrow \text{cov}[X,Y] = 0$

$$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$$ X, Y independent

---

Correlation

$$\rho(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{Var}[X]\,\mathrm{Var}[Y]}} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

Bivariate Normal Distribution

Independence

$$f_{XY}(x,y) = f(x,y) = f_X(x)\, f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma_x} \exp\left(-\frac{(x-\mu_x)^2}{2\sigma_x^2}\right)$$

$$\times \frac{1}{\sqrt{2\pi}\,\sigma_y} \exp\left(-\frac{(x-\mu_y)^2}{2\sigma_y^2}\right)$$

Bivariate Normal

Not independent   correlation ρ

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)} \times \left[\frac{(x-\mu_x)^2}{2\sigma_x^2} + \frac{(y-\mu_y)^2}{2\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right)$$