Prob Stats L08

Expectation

February 21, 2023

Expectation for Discrete R.V.

$f(a_i) = P(X = a_i)$

R.V. X pdf. $f : \{a_i\}_{i=1..n} \to \mathbb{R}$

$X \sim f$

$$\boxed{E[X] = \sum_{i=1}^{n} a_i \underbrace{\Pr(X = a_i)}_{w_i} = \sum_{i=1}^{n} a_i f(a_i)}$$

expectation of X

Average value that the R.V. X takes.

if $w_i = \frac{1}{n}$

$\Rightarrow E[X] = \frac{1}{n} \sum_{i=1}^{n} a_i$

$E[X] = \sum_{i=1}^{n} w_i \cdot a_i$

weights

$w_i \in [0,1]$

$\sum_{i=1}^{n} w_i = 1$

Example

$X \sim \text{Ber}(p)$ $\qquad X \in \{0, 1\}$

$\Pr[X=1] = p$

$\Pr[X=0] = 1-p$

$$f(x) = \begin{cases} p & \text{if } x=1 \\ 1-p & \text{if } x=0 \\ 0 & \text{o.w.} \end{cases}$$

$$E[X] = \sum_{i=0,1} i \cdot \Pr(X=i)$$

$$= 0 \cdot \Pr(X=0) + 1 \cdot \Pr(X=1)$$

$$= \cancel{0 \cdot (1-p)} + \cancel{1} \cdot (p)$$

$$= p$$

Example

$X \sim Geo(p)$ $\quad X \in \{\cancel{0}, 1, 2, \ldots\}$

non-neg integers

$$f_X(i) = (1-p)^{i-1}\, p$$

$$E[X] = \sum_{i=1}^{\infty} i \cdot f_X(i) = \sum_{i=1}^{\infty} i\,(1-p)^{i-1}\, p = \frac{1}{p}$$

$p = \frac{1}{2}$: $E[X] = \underline{1}\left(\frac{1}{2}\right) + \underline{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \underline{3}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) + \ldots$

$(1-p) = \frac{1}{2}$ $\quad + 2\left(\underline{\frac{1}{2}}\right)^2 + 3\left(\underline{\frac{1}{2}}\right)^3 + \ldots$

$E[X] = 2$

$$\sum_{i=1}^{\infty}\left(\frac{1}{2}\right)^i = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

$$= 1$$

$X \sim Geo(1/6)$

$\hookrightarrow E[X] = 6$

Die Roll $X \sim$ Die

$$f(x) = \begin{cases} \frac{1}{6} & x = \{1 \dots 6\} \\ 0 & \text{o.w.} \end{cases}$$

$$E[X] = \sum_{i=1}^{6} i \cdot \overbrace{\frac{1}{6}}^{\Pr(X=i)} \quad w_i$$

$$= \frac{1}{6} \sum_{i=1}^{6} i = \frac{1}{6}(21) = 3.5$$

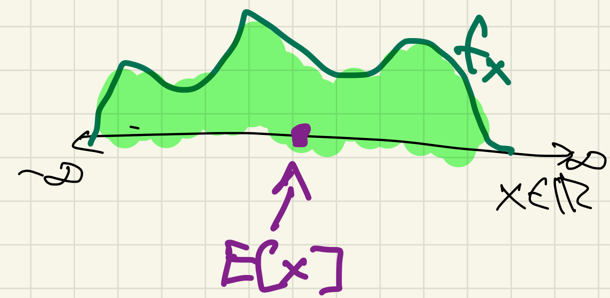
$$E[X] \qquad \cancel{x} \in \{1, \dots, 6\} \subset [1, 6]$$

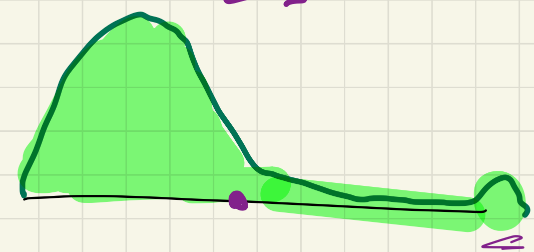
$$1 \le E[X] \le 6$$

Expectation on Continuous R.V.

$$X \sim f$$

$$\boxed{E[X] = \int_{x=-\infty}^{\infty} x\, f(x)\, dx}$$





$X \sim \mathrm{Exp}(\lambda)$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{o.w.} \end{cases}$$

$$E[X] = \int_{\cancel{-\infty}\ 0}^{\infty} x f(x)\, dx = \int_{x=0}^{\infty} x \lambda e^{-\lambda x}\, dx = \frac{1}{\lambda}$$

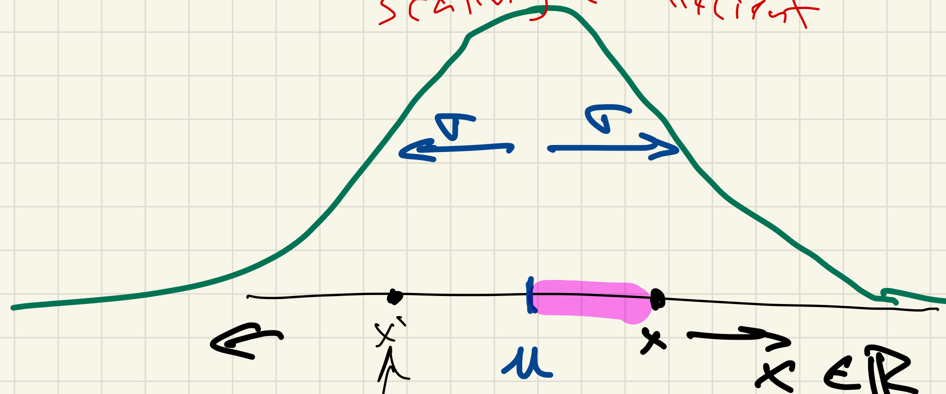
since $f(x) = 0$ $x < 0$

$X \sim N(\mu, \sigma^2)$

$E[X] = \mu$

$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$

scaling coefficient



$\mu - (x-\mu)$

$(x'-\mu)^2 = (x-\mu)^2$

$(\cancel{\mu} - (x-\mu) \cancel{-\mu})^2$

$(-(x-\mu))^2 = (x-\mu)^2$

Linearity of Expectation

2 Random Variables X, Y

new R.V. $Z = 2X + 3Y$

given $E[X] = 3.5$ $E[Y] = 2$

$$E[Z] = E[2X + 3Y]$$

$$= 2 \cdot E[X] + 3 \cdot E[Y]$$

$$= 2 \cdot (3.5) + 3(2)$$

$$= 7 + 6 = 13$$

- factor out const. multiple
  $E[\alpha X] = \alpha E[X]$
- decompose sums
  $E[X+Y] = E[X] + E[Y]$

Roll 10 fair die, add up results S

10 die $X_1, X_2, \ldots X_{10}$

$$S = \sum_{i=1}^{10} X_i \qquad E[X_i] = 3.5$$

$$E[S] = E[X_1 + X_2 \cdots + X_{10}]$$

$$= E[X_1] + E[X_2] + \cdots + E[X_{10}]$$

$$= 10 \cdot E[X_i] = 10 \cdot (3.5) = 35$$

$X \sim \text{Bin}(n, p)$

Binomial

distribution of number of successes of n trials, each w.p. p.

$E[X]$

$X = \sum_{i=1}^{n} B_i$

$B_i \sim \text{Ber}(p)$

$E[B_i] = p$

$= E[X] = E\left[\sum_{i=1}^{n} B_i\right]$

$= \sum_{i=1}^{n} E[B_i] = \sum_{i=1}^{n} p$

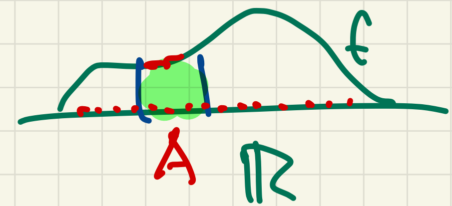
$= n \cdot p$

$X \sim$	$E[X]$
$\mathrm{Ber}(p)$	p
$\mathrm{Geo}(p)$	$1/p$
$\mathrm{Bin}(n,p)$	$n \cdot p$
$\mathrm{Exp}(\lambda)$	$1/\lambda$
$N(\mu, \sigma^2)$	μ

Discrete RV. X

$$f_X(a) = Pr(X=a)$$

Contin RV. X



$$f_X(a) \neq Pr(X=a)$$

$X \sim Unif(\alpha, \beta)$