Set Laws & Probability

CS 3130/ECE 3530:
Probability and Statistics for Engineers

Jan 17, 2023

Sets

Sample Space Ω

subsets $A \subseteq \Omega$

event

A, B : union $A \cup B$ "or" (or both)

intersection $A \cap B$ "and"

Commutative Law

For two sets A, B the **Commutative Law** holds that

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

$$A \cap B = B \cap A$$

$$A - B \neq B - A$$

Associative Law

For three sets A, B, C the **Associative Law** holds that

$$(A \cup B) \cup C = A \cup \boxed{(B \cup C)}$$

$$(A \cap B) \cap C = A \cap \underline{(B \cap C)}$$

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For three sets A, B, C the **Associative Law** holds that

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

Example:
$A = \{3, 4, 5, 6\}$
$B = \{1, 3, 6\}$
$C = \{3, 5\}$
What is $(A \cup B) \cup C$?

$A \cup B = \{1, 3, 4, 5, 6\}$

$(A \cup B) \cup C = \{1, 3, 4, 5, 6\}$

$(B \cup C) = \{1, 3, 5, 6\}$

$A \cup (B \cup C) = \{1, 3, 4, 5, 6\}$

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What is $(A \cup B) \cup C$?
What is $(A \cap B) \cap C$?

? $(A \cup B) \cap C =$

$\longrightarrow$

$A \cap B = \{3, 6\}$
$(A \cap B) \cap C = \{3\}$

Distributive Law

For three sets A, B, C the **Distributive Law** holds that

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

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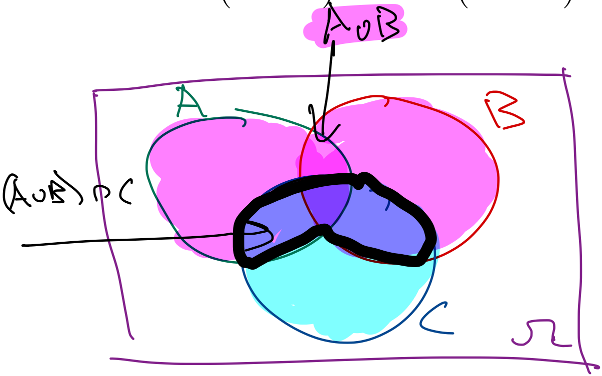

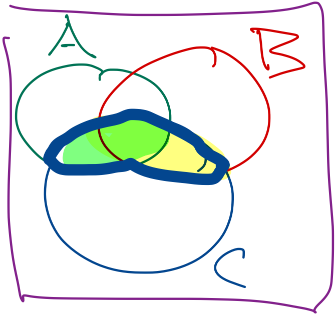

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DeMorgan's Law

Complement of union or intersection:

$$(A \cup B)^c = A^c \cap B^c$$

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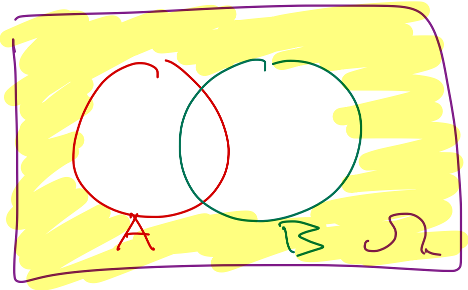

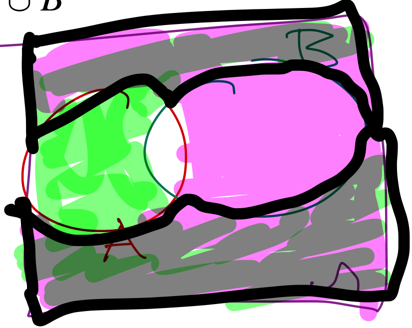

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What is the English translation for both sides of the equations above?

If something is not in both A and B

then it either not in A or not in B.

Exercises

Check whether the following statements are true or false. (Hint: you might use Venn diagrams.)

- $A - B \subseteq A$ True
- $(A - B)^c = A^c \cup B$ True
- $A \cup B \subseteq B$ Not always, unless $\boxed{A \subseteq B}$
- $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ True

$$(A-B = \boxed{A \cap B^c)^c} \Longrightarrow A^c \cup B$$

DeMorgans

Probability

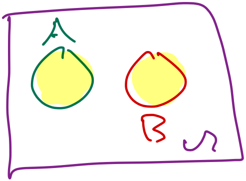
$P : \{A \subseteq \Omega\} \to [0,1]$

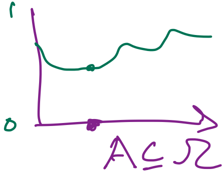
Definition

A **probability function** on a finite sample space Ω assigns every event $A \subseteq \Omega$ a number in $[0, 1]$, such that

1. $P(\Omega) = 1$
2. $P(A \cup B) = P(A) + P(B)$ when $\boxed{A \cap B = \emptyset}$

$P(A)$ is the **probability** that event A occurs.





Equally Likely Outcomes

The number of elements in a set A is denoted $|A|$.

cardinality of A : $|A|$

"size" or

$\#$ elements in A

Equally Likely Outcomes

$|A| \leq |\Omega|$

$|A| \geq 0$

The number of elements in a set A is denoted $|A|$.

If Ω has a finite number of elements, and each is equally likely, then the probability function is given by

$$P(A) = \frac{|A|}{|\Omega|} \stackrel{?}{=} \frac{2}{10} = 0.2$$

$\in [0, 1]$

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Example: Rolling a 6-sided die $|\Omega| = 6$

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Example: Rolling a 6-sided die

- $P(\{1\}) = 1/6$
- $P(\{1, 2, 3\}) = 1/2$ $= \frac{3}{6}$

Repeated Experiments

If we do two runs of an experiment with sample space Ω, then we get a new experiment with sample space

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Properties:
Order matters: $(1, 2) \neq (2, 1)$
Repeats are possible: $(1, 1) \in \mathbb{N} \times \mathbb{N}$

More Repeats

Repeating an experiment n times gives the sample space

$$\mathbb{R}^d$$

$$v = (v_1, v_2, \ldots, v_d)$$

$$\begin{aligned}\Omega^n &= \Omega \times \cdots \times \Omega \quad (n \text{ times}) \\ &= \{(x_1, x_2, \ldots, x_n) : x_i \in \Omega \text{ for all } i\}\end{aligned}$$

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The element $(x_1, x_2, \ldots, x_n)$ is called an n-**tuple**.

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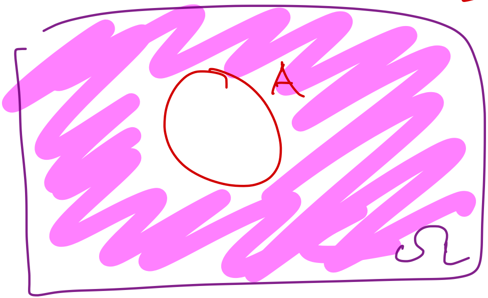
If $|\Omega| = k$, then $|\Omega^n| = k^n$.

Probability Rules

Probability Rules

Complement of an event A:

$$P(A^c) = 1 - \underbrace{P(A)}_{\in [0,1]}$$



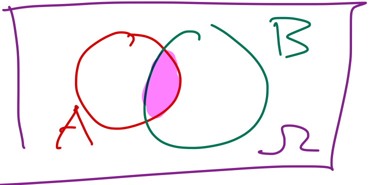
Probability Rules

Complement of an event A:

$$P(A^c) = 1 - P(A)$$

Union of two overlapping events $A \cap B \neq \emptyset$:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



Exercise

You are picking a number out of a hat, which contains the numbers 1 through 100. What are the following events and their probabilities?

- The number has a single digit $9/100$
- The number has two digits $\frac{90}{100} = \frac{9}{10}$
- A The number is a multiple of 4 $= \frac{25}{100} = \frac{1}{4}$
- The number is not a multiple of 4 $= \frac{3}{4}$ $= Pr(A^c) = 1 - Pr(A) = \frac{3}{4}$
- The sum of the number's digits is 5

$\frac{6}{100} = \frac{|\{5, 14, 23, 32, 41, 50\}|}{100}$

Permutations

A **permutation** is an ordering of an n-tuple. For instance, the n-tuple $(1, 2, 3)$ has the following permutations:

$$(1, 2, 3), (1, 3, 2), (2, 1, 3)$$
$$(2, 3, 1), (3, 1, 2), (3, 2, 1)$$

deck of cards

finish order of race.

anagram

dog ~ god

~ ogd

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$$(2, 3, 1), (3, 1, 2), (3, 2, 1)$$

The number of unique orderings of an n-tuple is n **factorial**:

$$\underline{n!} = n \times (n - 1) \times (n - 2) \times \cdots \times 2 \times 1$$

Permutations

$\underbrace{n \cdot (n-1) \cdot (n-2) \cdots \boxed{(n/2)}}_{n/2+1} \cdot (n/2-1) \cdots 1$

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$$(2,3,1), (3,1,2), (3,2,1)$$

$\boxed{n!}$ or $\left(\frac{n}{2}\right)^{\left(\frac{n}{2}\right)}$

larger

The number of unique orderings of an n-tuple is n **factorial**:

$\left(\frac{n}{2}\right)^{(n/2)} < n! \leq n^n$

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2$$

How many ways can you rearrange $(1, 2, 3, 4)$?

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

Exercise

Consider 4 balls in an urn, with labels A, B, C, and D. Consider I select them out of the urn (without replacement) one at a time.

- What is the probability I pick them out in order (A, B, C, D)? $\Rightarrow Pr(\ldots) = 1/24 = \frac{1}{4!}$

Exercise

Consider 4 balls in an urn, with labels A, B, C, and D. Consider I select them out of the urn (without replacement) one at a time.

- What is the probability I pick them out in order (A, B, C, D)?
- What is the probability I pick them out in order (B, C, A, D)? $= \frac{1}{24}$

Exercise

Consider 4 balls in an urn, with labels A, B, C, and D. Consider I select them out of the urn (without replacement) one at a time.

- What is the probability I pick them out in order (A, B, C, D)?
- What is the probability I pick them out in order (B, C, A, D)?
- What is the probability that the last element chosen is A?

? $\frac{4}{24}$, ? $\frac{6}{24}$? $\frac{1}{24}$

$\frac{1}{4}$ =

Exercise

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- What is the probability I pick them out in order (A, B, C, D)?
- What is the probability I pick them out in order (B, C, A, D)?
- What is the probability that the last element chosen is A?
- What is the probability that the last element chosen is D?