Set Laws & Probability

CS 3130/ECE 3530:
Probability and Statistics for Engineers

Jan 17, 2023

Commutative Law

For two sets A, B the **Commutative Law** holds that

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

Associative Law

For three sets A, B, C the **Associative Law** holds that

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$C = \{3, 5\}$
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DeMorgan's Law

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What is the English translation for both sides of the equations above?

Exercises

Check whether the following statements are true or false. (Hint: you might use Venn diagrams.)

- $A - B \subseteq A$
- $(A - B)^c = A^c \cup B$
- $A \cup B \subseteq B$
- $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

Probability

Definition

A **probability function** on a finite sample space Ω assigns every event $A \subseteq \Omega$ a number in $[0, 1]$, such that

1. $P(\Omega) = 1$
2. $P(A \cup B) = P(A) + P(B)$ when $A \cap B = \emptyset$

$P(A)$ is the **probability** that event A occurs.

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- $P(\{1\}) = 1/6$
- $P(\{1, 2, 3\}) = 1/2$

Repeated Experiments

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Properties:
Order matters: $(1, 2) \neq (2, 1)$
Repeats are possible: $(1, 1) \in \mathbb{N} \times \mathbb{N}$

More Repeats

Repeating an experiment n times gives the sample space

$$
\begin{aligned}
\Omega^n &= \Omega \times \cdots \times \Omega \quad (n \text{ times}) \\
&= \{(x_1, x_2, \ldots, x_n) : x_i \in \Omega \text{ for all } i\}
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If $|\Omega| = k$, then $|\Omega^n| = k^n$.

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Union of two overlapping events $A \cap B \neq \emptyset$:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Exercise

You are picking a number out of a hat, which contains the numbers 1 through 100. What are the following events and their probabilities?

- The number has a single digit
- The number has two digits
- The number is a multiple of 4
- The number is not a multiple of 4
- The sum of the number's digits is 5

Permutations

A **permutation** is an ordering of an n-tuple. For instance, the n-tuple $(1, 2, 3)$ has the following permutations:

$$(1, 2, 3), (1, 3, 2), (2, 1, 3)$$
$$(2, 3, 1), (3, 1, 2), (3, 2, 1)$$

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How many ways can you rearrange $(1, 2, 3, 4)$?

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Consider 4 balls in an urn, with labels A, B, C, and D. Consider I select them out of the urn (without replacement) one at a time.

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- What is the probability that the last element chosen is D?