An Analytical Approach to Single Scattering for Anisotropic Media and Light Distributions

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Outline

- Introduction
- Related Work
- An Analytical Approach to Single Scattering
  - Air-Light Integral
  - Simplified Formulation
  - Analytical Reformulations
  - Combined Formulation
- Implementation
- Results
- Discussion and Future Work
- Conclusion

Introduction

- Applications
  - Movie and gaming industries
  - Industrial design & safety-oriented research
- Motivation
  - Intricacy of RTE $\Rightarrow$ numerical methods
  - Air-light integral $\Rightarrow$ analytical approach

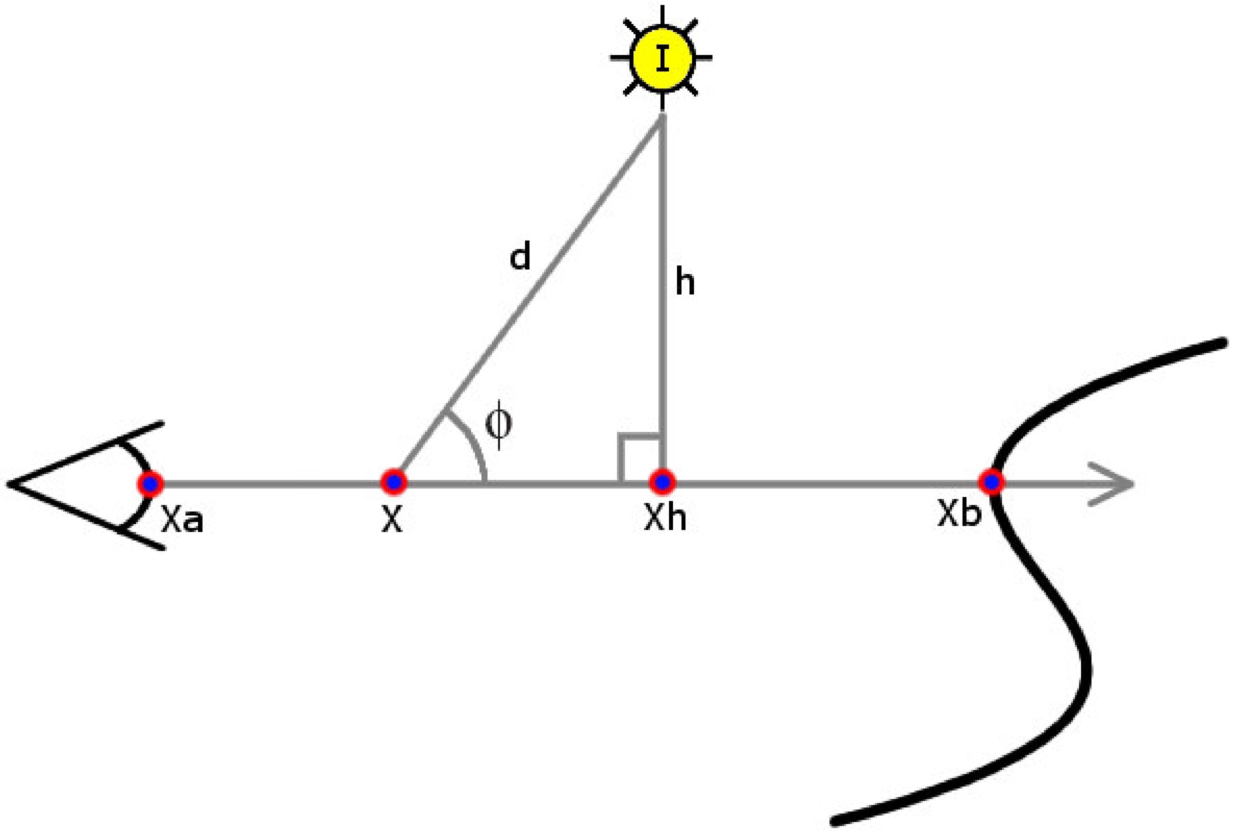
Related Work

- Numerical Ray-Marching / Volume-Slicing
  - Riemann sums $\Rightarrow$ under-sampling artifacts

- Analytical Approaches
  - Promising alternative
  - Pegoraro et al. EG09
    - Isotropic light sources
    - Slow convergence of phase function series
    - Loss of efficiency in optically thick media
    - CPU implementation

Analytical Single Scattering

- Air-Light Integral



Analytical Single Scattering

- Air-Light Integral
  - Definition

$$L(x_a, \vec{\omega}) = e^{-\kappa_t (x_b - x_a)} L_b(x_b, \vec{\omega}) + L_m(x_a, x_b, \vec{\omega})$$

  - Medium Radiance

$$L_m(x_a, x_b, \vec{\omega}) = \kappa_s e^{\kappa_t x_a} \int_{x_a}^{x_b} \frac{e^{-\kappa_t \left(x + \sqrt{h^2 + (x - x_h)^2}\right)}}{h^2 + (x - x_h)^2}$$

$$I\left(\arccos\left(\frac{d_{el} x + d_{lel}}{\sqrt{h^2 + (x - x_h)^2}}\right)\right) \Phi\left(\arctan\left(\frac{x - x_h}{h}\right) + \frac{\pi}{2}\right) dx$$

Analytical Single Scattering

- Simplified Formulation
  - Change of Variable

$$u = \frac{x - x_h}{h} \text{ with } u \in (-\infty, 0, \infty)$$

  - Medium Radiance

$$L_m(x_a, x_b, \vec{\omega}) = \frac{\kappa_s}{h} e^{\kappa_t (x_a - x_h)} \int_{\frac{x_a - x_h}{h}}^{\frac{x_b - x_h}{h}} \frac{e^{-H(u + \sqrt{1+u^2})}}{1+u^2}$$

$$I_c\left(\frac{d_{el} u + d_c}{\sqrt{1+u^2}}\right) \Phi_c\left(-\frac{u}{\sqrt{1+u^2}}\right) du$$

Analytical Single Scattering

- First Analytical Reformulation
  - Change of Variable

$$v = u + \sqrt{1+u^2} \text{ where } v \in (0, 1, \infty)$$

  - Medium Radiance

$$L_m(x_a, x_b, \vec{\omega}) = \frac{\kappa_s}{h} e^{\kappa_t (x_a - x_h)} 2 \int_{v_a}^{v_b} \frac{e^{-Hv}}{v^2+1}$$

$$I_c\left(\frac{d_{el}(v^2-1)+2d_c v}{v^2+1}\right)\Phi_c\left(-\frac{v^2-1}{v^2+1}\right)\, dv$$

Analytical Single Scattering

- First Analytical Reformulation
  - Taylor Series Expansion

$$L_m(x_a, x_b, \vec{\omega}) = \frac{\kappa_s}{h} e^{\kappa_t (x_a - x_h)} 2 \sum_{n=0}^{N-1} c_n \int_{v_a}^{v_b} \frac{e^{-Hv}}{v^2+1} v^n \, dv$$

  - Analytical Solution

$$L_m(x_a, x_b, \vec{\omega}) = \frac{\kappa_s}{h} e^{\kappa_t (x_a - x_h)} 2 \sum_{n=0}^{N-1} c_n \Bigg((-1)^{\lfloor \frac{n}{2} \rfloor} I_{(n \bmod 2)}(-H, v_a, v_b) + \sum_{j=0}^{n-2} \left(e^{-Hv_b} v_b^j - e^{-Hv_a} v_a^j \right) c(-H, j, n) \Bigg)$$

Analytical Single Scattering

- First Analytical Reformulation
  - Constants

$$c(a,j,n) = \sum_{\substack{i=(n-2-j) \bmod 2 \\ i+=2}}^{n-2-j} (-1)^{\frac{n-i-j}{2}} \frac{(i+j)!}{j!} \left(-\frac{1}{a}\right)^{i+1}$$

  - Functions

$$I_0(a,v_a,v_b) = i_0(a,v_b) - i_0(a,v_a)$$
$$I_1(a,v_a,v_b) = i_1(a,v_b) - i_1(a,v_a)$$

$$i_0(a,v) = \sin(a)\Re(Ei(av+\imath a)) - \cos(a)\Im(Ei(av+\imath a))$$
$$i_1(a,v) = \cos(a)\Re(Ei(av+\imath a)) + \sin(a)\Im(Ei(av+\imath a))$$

Analytical Single Scattering

- Second Analytical Reformulation
  - Change of Variable

$$w = u - \sqrt{1+u^2} \text{ with } w \in (-\infty, -1, 0)$$

  - Medium Radiance

$$L_m(x_a, x_b, \vec{\omega}) = \frac{\kappa_s}{h} e^{\kappa_t (x_a - x_h)} 2 \int_{w_a}^{w_b} \frac{e^{\frac{H}{w}}}{w^2+1}$$

$$I_c\left(-\frac{d_{el}(w^2-1)+2d_c w}{w^2+1}\right) \Phi_c\left(\frac{w^2-1}{w^2+1}\right) \, dw$$

Analytical Single Scattering

- Second Analytical Reformulation
  - Taylor Series Expansion

$$L_m(x_a, x_b, \vec{\omega}) = \frac{\kappa_s}{h} e^{\kappa_t (x_a - x_h)} 2 \sum_{n=0}^{N-1} d_n \int_{w_a}^{w_b} \frac{e^{\frac{H}{w}}}{w^2 + 1} w^n \, dw$$

  - Analytical Solution

$$L_m(x_a, x_b, \vec{\omega}) = \frac{\kappa_s}{h} e^{\kappa_t (x_a - x_h)} 2 \sum_{n=0}^{N-1} d_n \Bigg((-1)^{\lfloor \frac{n}{2} \rfloor} J_{(n \bmod 2)}(H, w_a, w_b)$$

$$+ J_e(H, w_a, w_b) \, H \, d(H, 0, n) - \sum_{j=0}^{n-2} \left(e^{\frac{H}{w_b}} w_b^{j+1} - e^{\frac{H}{w_a}} w_a^{j+1} \right) d(H, j, n) \Bigg)$$

Analytical Single Scattering

- Second Analytical Reformulation
  - Constants

$$d(a,j,n) = \sum_{\substack{i=(n-2-j) \bmod 2 \\ i+=2}}^{n-2-j} (-1)^{\frac{n-i-j}{2}} \frac{j!}{(i+j+1)!} a^i$$

  - Functions

$$J_0(a,w_a,w_b) = j_0(a,w_b) - j_0(a,w_a)$$

$$J_1(a,w_a,w_b) = j_1(a,w_b) - j_1(a,w_a)$$

$$J_e(a,w_a,w_b) = Ei\left(\frac{a}{w_b}\right) - Ei\left(\frac{a}{w_a}\right)$$

$$j_0(a,w) = -\sin(a)\Re\left(Ei\left(\frac{a}{w}+\imath a\right)\right) + \cos(a)\Im\left(Ei\left(\frac{a}{w}+\imath a\right)\right)$$

$$j_1(a,w) = \cos(a)\Re\left(Ei\left(\frac{a}{w}+\imath a\right)\right) + \sin(a)\Im\left(Ei\left(\frac{a}{w}+\imath a\right)\right) - Ei\left(\frac{a}{w}\right)$$

Analytical Single Scattering

- Combined Formulation

$$L_m(x_a, x_b, \vec{\omega}) = L_m(x_h, x_b, \vec{\omega}) - L_m(x_h, x_a, \vec{\omega}) = \frac{\kappa_s}{h} e^{\kappa_t (x_a - x_h)}$$

$$\left(\int_0^{\frac{x_b - x_h}{h}} \frac{e^{-H(u+\sqrt{1+u^2})}}{1+u^2} I_c \left(\frac{d_{el} u + d_c}{\sqrt{1+u^2}} \right) \Phi_c \left(-\frac{u}{\sqrt{1+u^2}} \right) du - \right.$$

$$\left. \int_0^{\frac{x_a - x_h}{h}} \frac{e^{-H(u+\sqrt{1+u^2})}}{1+u^2} I_c \left(\frac{d_{el} u + d_c}{\sqrt{1+u^2}} \right) \Phi_c \left(-\frac{u}{\sqrt{1+u^2}} \right) du \right)$$

  - Trivial correspondence between expansions
  - Expansion in mid-point of integration domain

Implementation

- Complex-Valued Exponential Integral
  - Power Series

$$Ei(z) = \gamma + \ln(z) + \sum_{k=1}^{\infty} \frac{z^k}{k\,k!}$$

  - Asymptotic Series

$$Ei(z) \approx e^z \sum_{k=1}^{K} \frac{k!}{k\,z^k} + \imath\ sign(\Im(z))\pi$$

  - Criterion

$$1 \leq \rho \quad \wedge \quad x - \lfloor \rho \rfloor < ln(\varepsilon) - 1$$

Implementation

- Real-Valued Exponential Integral
  - Polynomial Approximation: $0 < x \leq 1$

$$E_1(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 - ln(x) + \varepsilon(x)$$

  - Rational Approximation: $1 < x < \infty$

$$E_1(x) = \frac{e^{-x}}{x}\left(\frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + x^4}{b_0 + b_1 x + b_2 x^2 + b_3 x^3 + x^4} + \varepsilon(x)\right)$$

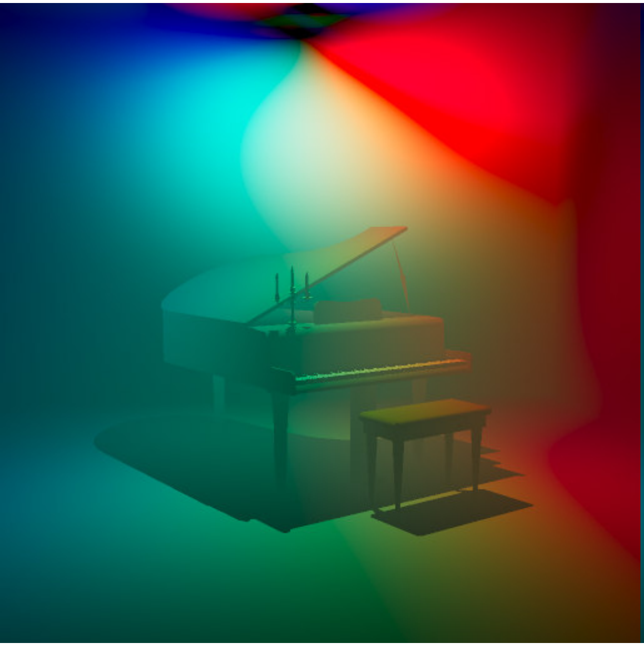
Results

A bench in a park covered in anisotropic mist

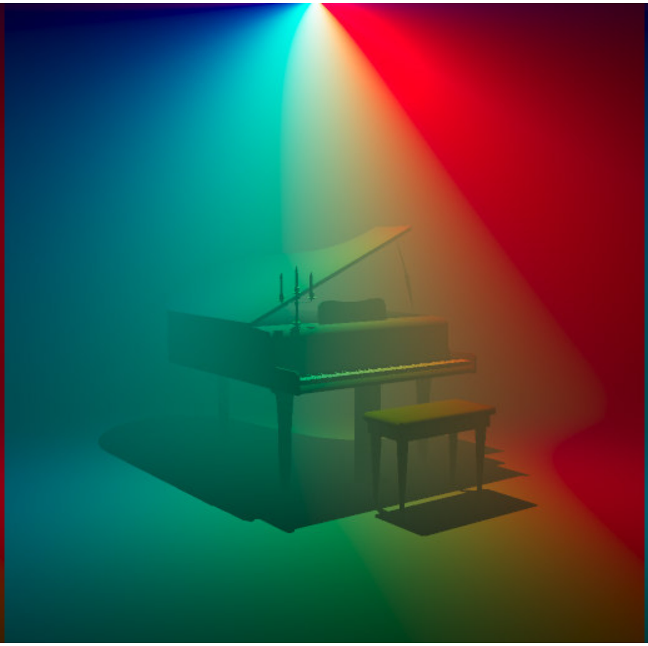
First Formulation Dual-Formulation Monte Carlo

Results

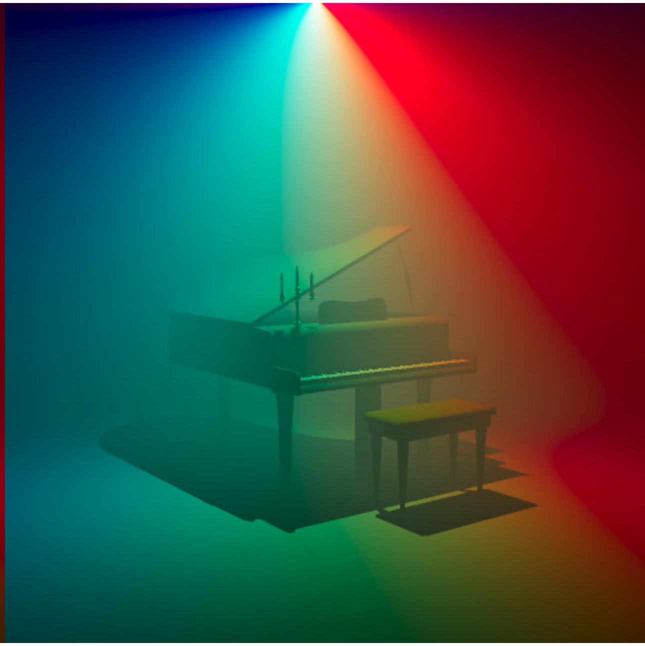
A concert stage lit by 3 colorful spotlights



First Formulation



Dual-Formulation



Monte Carlo

Results

A foggy road illuminated by anisotropic lights

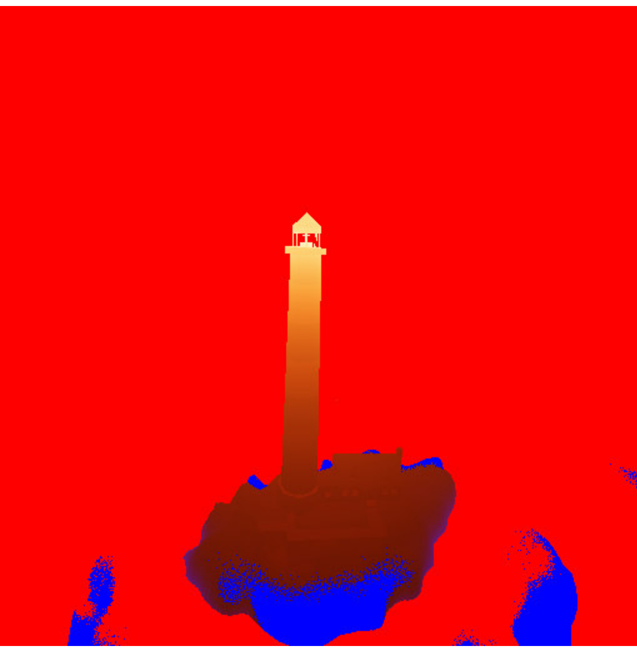
Fixed Point

Mid-Point

Monte Carlo

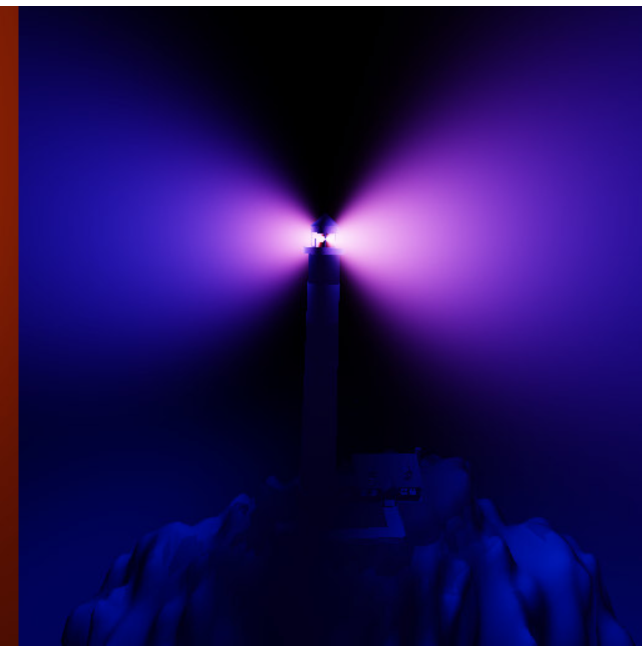
Results

A lighthouse in thick brume rendered in real-time



Power Series

Power/Asymptotic



Power/Asymptotic

Discussion and Future Work

- Better representation of anisotropy
- Volumetric shadows
- Inhomogeneous media
- Impact on surface shading

Conclusion

- Novel analytical approach
- Dual-formulation via domain-partitioning
- Increased accuracy in anisotropic media
- Handles anisotropic lights analytically
- Neither pre-computation nor storage
- Practical implementation
- Stable and efficient evaluations
- Real-time performance

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- Thank you !