Non-Euclidean Geometry for SAT

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A set of Matlab functions have been created to allow the exploration of the usefulness of setting geometric SAT into non-Euclidean geometry. Three major models are represented: (1) the Poincare half-plane (PH), (2) the Poincare disk (PD), and (3) the Beltrami-Klein disk (BK). For more information on these formulations, see Appendixes A, B and C, respectively (taken from Wikipedia).

The Poincare Half-Plane ($\mathbb{H}$)

Since the goal is to represent n-dimensional polytopes which represent the feasible region for a SAT solution, it is necessary to represent these in non-Euclidean spaces. Examples are given in 2D for illustration purposes. Consider the knowledge base with the single clause: $A \vee \neg B$. Then in regular Euclidean space, the feasible region (after chopping the (0,1) vertex) will be the triangle [(0,0), (1,0), (1,1)] (see Figure 1).

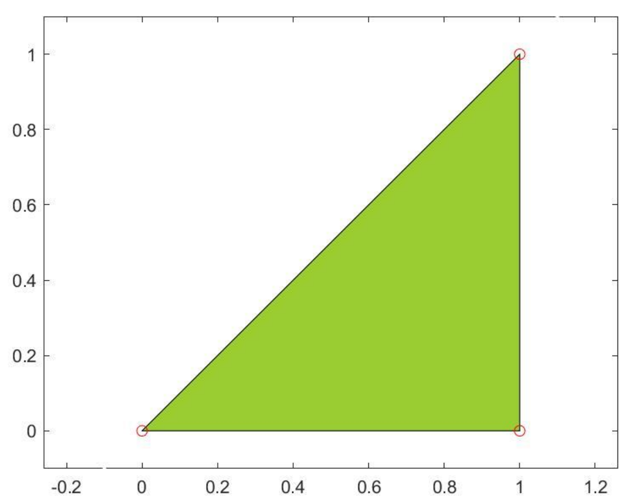


Figure 1. Feasible Region for KB with Clause $A \vee \neg B$.

Converting this to the Poincare half-plane model requires deciding how the unit square will be represented. The most straightforward is to use the same points: (0,0), (1,0), (1,1), (0,1); however, the points on the x-axis are not in $\mathbb{H}$, and this poses some problems. Figure 2 shows how the unit square transforms into $\mathbb{H}$.

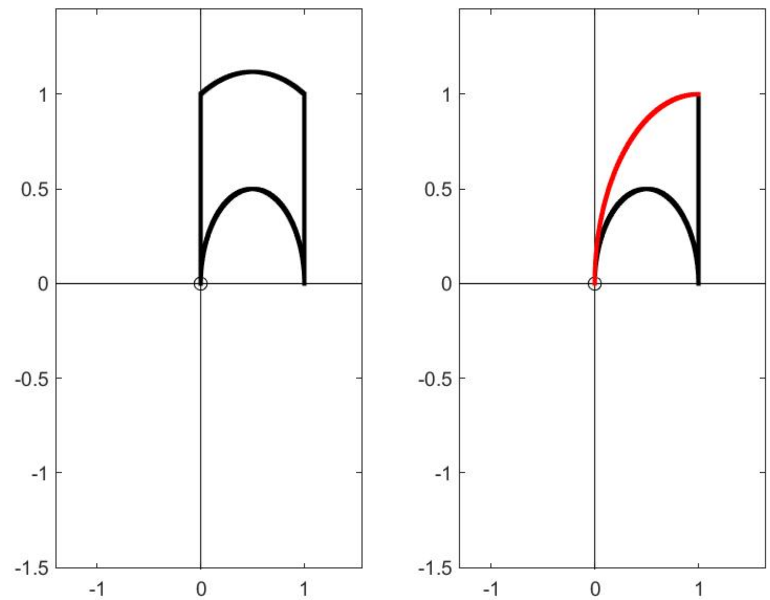


Figure 2. (left) Unit Square in Poincare Half-Plane. (right) Feasible Region for KB with Clause $A \vee \neg B$ (cutting plane shown in red).

The left side plot is achieved as follows; first get the sides of the unit square:

seg1 = NON_PH_seg_pts([0,0],[1,0]);

seg2 = NON_PH_seg_pts([1,0],[1,1]);

seg3 = NON_PH_seg_pts([1,1],[0,1]);

seg4 = NON_PH_seg_pts([0,1],[0,0]);

Then plot them:

NON_plot_PH_pts([seg1;seg2;seg3;seg4],1,'k.');

The right side is found by first finding the cutting line, then plotting the three remaining segments:

seg8 = NON_PH_seg_pts([1,1],[0,0]);

NON_plot_PH_pts([seg1;seg2],1,'k.');

NON_plot_PH_pts(seg8,1,'r.');

Note that the area of the resulting triangle is $\frac{\pi}{2}$, so that the area of the unit square in $\mathbb{H}$ is not 1!

The Poincare Disk (𝔻)

The unit square represented by [(0,0),(0.5,0),(0.5,0.5),(0,0.5)] is shown on the left side of Figure 3, while the same feasible region is shown on the right side of the figure.

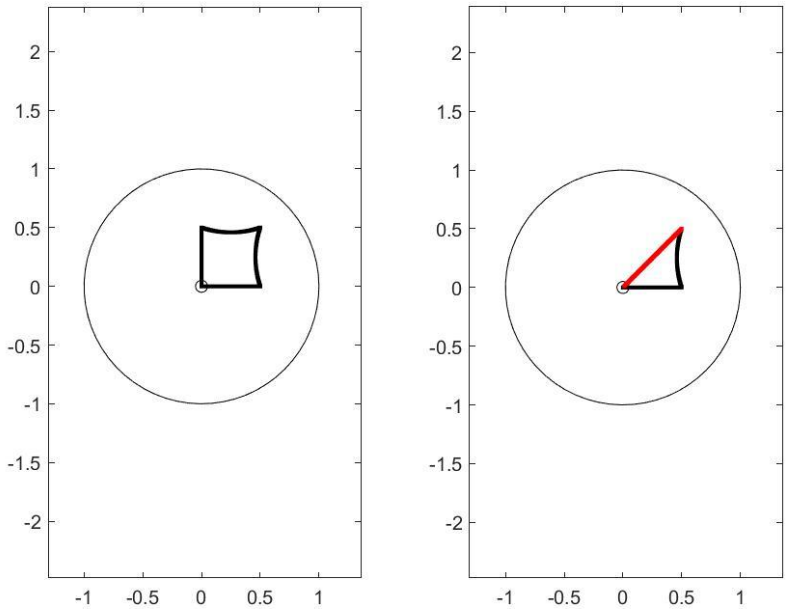


Figure 3. (left) Unit Square in Poincare Disk. (right) Feasible Region for KB with Clause $A \vee \neg B$ (cutting plane shown in red).

This is produced as follows; for the left side:

```
seg1 = NON_PD_seg_pts([0,0],[0.5,0]);

seg2 = NON_PD_seg_pts([0.5,0],[0.5,0.5]);

seg3 = NON_PD_seg_pts([0.5,0.5],[0,0.5]);

seg4 = NON_PD_seg_pts([0,0.5],[0,0]);

NON_plot_PD_pts([seg1;seg2;seg3;seg4],1,'k.');
```

The figure on the right:

```
seg5 = NON_PD_seg_pts([0.5,0.5],[0,0]);

NON_plot_PD_pts([seg1;seg2],1,'k.');

>> NON_plot_PD_pts(seg5,1,'r.');
```

Converting Points between Representations

Sometimes it is convenient to change representation; therefore, we have provided functions to convert as follows:

- NON_H2D: Poincare half-plane to Poincare disk
- NON_D2H: Poincare disk to Poincare half-plane
- NON_H2K: Poincare half-plane to Beltrami-Klein
- NON_K2H: Beltrami-Klein to Poincare half-plane
- NON_D2K: Poincare disk to Beltrami-Klein
- NON_K2D: Beltrami-Klein to Poincare disk

Note that all of these take on complex number input and produce one complex number output.

Distance Between Points

Functions have been provided to compute the distance between points:

- NON_norm_PD: Poincare norm
- NON_norm_PH: Poincare norm
- NON_norm_BK: Beltrami-Klein norm

Possible Representation of the Hypercube in n-D

A possible representation of the hypercube in n-D is to project the corners of the unit cube (centered a 0 and scaled to circumscribe the unit sphere) onto the unit hypersphere in $\mathbb{D}$. Figure 4 shows this for 2D; note that the corners of the square are ideal points (see Appendix D), and not in $\mathbb{D}$.

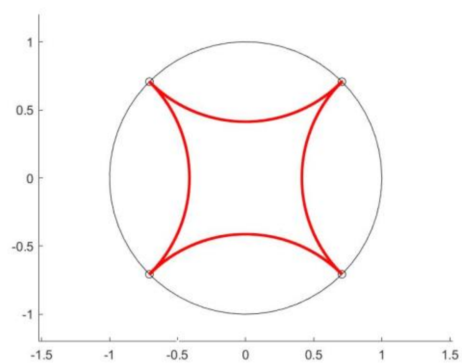


Figure 4. A projection of the Circumscribed Hypercube onto the Unit Sphere. In this case, the corners are not in $\mathbb{D}$, but rather are ideal points on the circle boundary.

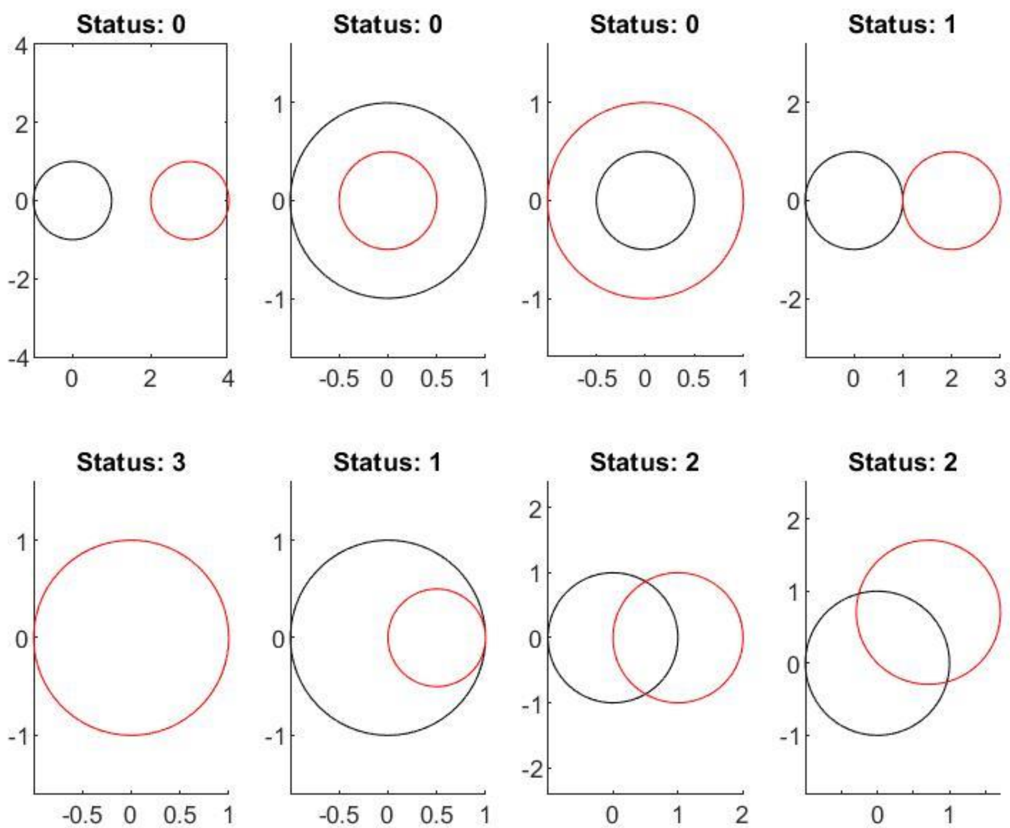
This makes a shape that is geometrically similar to the square, but note that its area is π. Figure 4 is produced by:

NON_H2circumcribedinPD; % files in PSSAT/non_Euclidean/develop

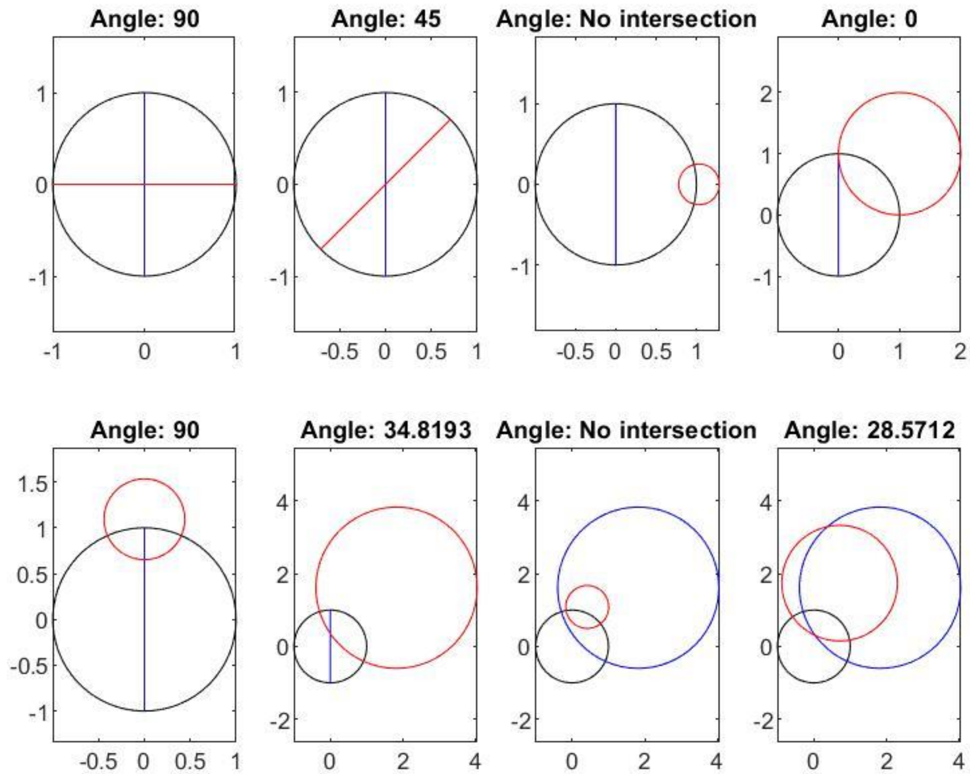
27 June 2023

Development over last few weeks is in PSSAT/non_Euclidean/ with prefix NE_ (see Non-Euclidean-Matlab-Functions.pdf and NE_Function_Dependencies.pdf).

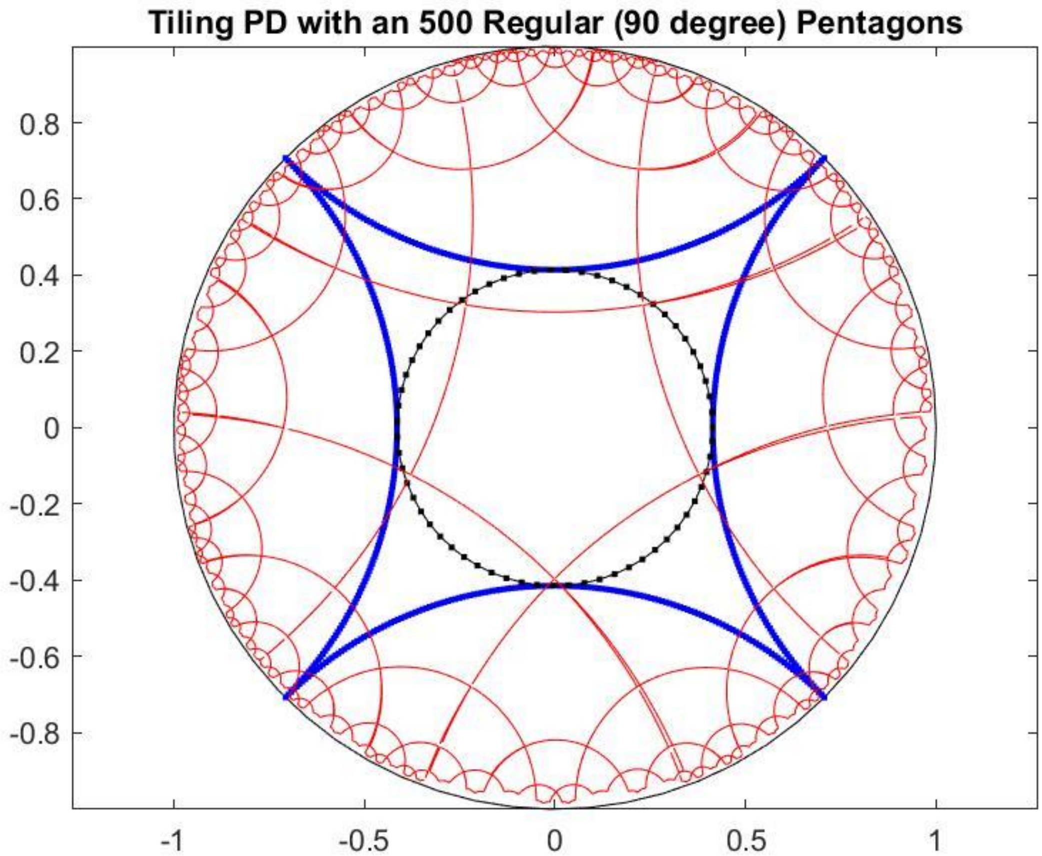
Results testing Euclidean circle intersection (NE_test_int_E2_2circles):



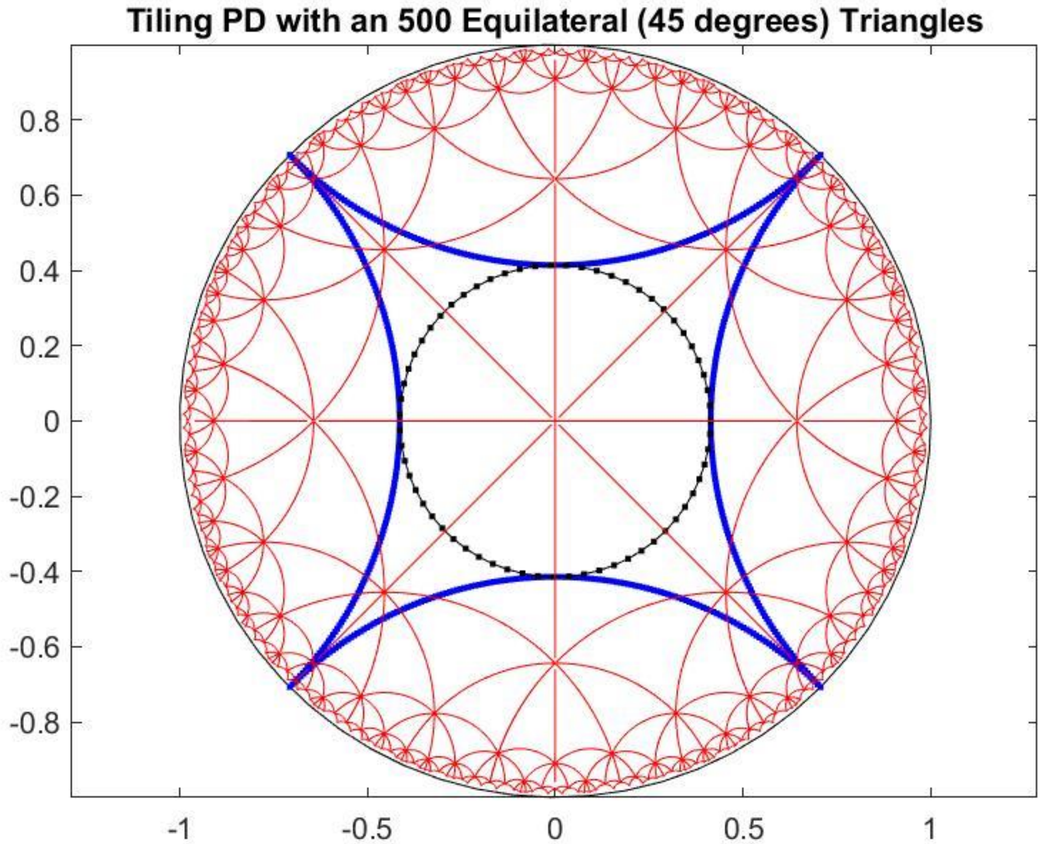
Results testing great circle intersection angles (NE_test_angle_2circles):



Results tiling PD with a regular pentagon (NE_inversion_experiment1):



Results tiling PD with 45-degree angle regular triangle (NE_inversion_experiment2):


Tiling PD with an 500 Equilateral (45 degrees) Triangles
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-0.5
0
0.5
1

Appendix A: Poincare Half-Plane

Poincaré half-plane model

文A 7 languages

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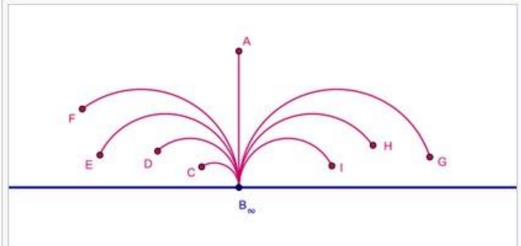
In non-Euclidean geometry, the **Poincaré half-plane model** is the upper half-plane, denoted below as $\mathbf{H} = \{\langle x, y \rangle \mid y > 0; x, y \in \mathbb{R}\}$, together with a metric, the Poincaré metric, that makes it a model of two-dimensional hyperbolic geometry.



Parallel rays in Poincare half-plane model of hyperbolic geometry

Equivalently the Poincaré half-plane model is sometimes described as a complex plane where the imaginary part (the *y* coordinate mentioned above) is positive.

The Poincaré half-plane model is named after Henri Poincaré, but it originated with Eugenio Beltrami who used it, along with the Klein model and the Poincaré disk model, to show that hyperbolic geometry was equiconsistent with Euclidean geometry.

This model is conformal which means that the angles measured at a point are the same in the model as they are in the actual hyperbolic plane.

The Cayley transform provides an isometry between the half-plane model and the Poincaré disk model.

This model can be generalized to model an $n + 1$ dimensional hyperbolic space by replacing the real number *x* by a vector in an *n* dimensional Euclidean vector space.

Metric [edit]

The metric of the model on the half-plane, $\{\langle x, y \rangle \mid y > 0\}$, is:

$$(ds)^2 = \frac{(dx)^2 + (dy)^2}{y^2}$$

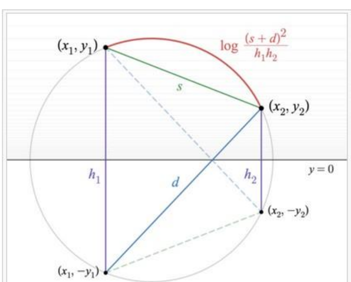
where *s* measures the length along a (possibly curved) line. The ***straight lines*** in the hyperbolic plane (geodesics for this metric tensor, i.e., curves which minimize the distance) are represented in this model by circular arcs perpendicular to the *x*-axis (half-circles whose centers are on the *x*-axis) and straight vertical rays perpendicular to the *x*-axis.

Distance calculation [edit]

If $p_1 = \langle x_1, y_1 \rangle$ and $p_2 = \langle x_2, y_2 \rangle$ are two points in the half-plane $y > 0$ and $\tilde{p}_1 = \langle x_1, -y_1 \rangle$ is the reflection of p_1 across the *x*-axis into the lower half plane, the *distance* between the two points under the hyperbolic-plane metric is:

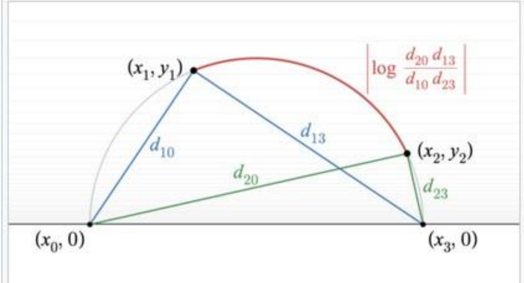
$$\begin{aligned} \operatorname{dist}(p_1, p_2) &= 2 \operatorname{arsinh} \frac{\|p_2 - p_1\|}{2\sqrt{y_1 y_2}} \\ &= 2 \operatorname{artanh} \frac{\|p_2 - p_1\|}{\|p_2 - \tilde{p}_1\|} \\ &= 2 \ln \frac{\|p_2 - p_1\| + \|p_2 - \tilde{p}_1\|}{2\sqrt{y_1 y_2}}, \end{aligned}$$

where $\|p_2 - p_1\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ is the Euclidean distance between points p_1 and p_2, $\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2 + 1}\right)$ is the inverse hyperbolic sine, and $\operatorname{artanh} x = \frac{1}{2} \ln((1 + x)/(1 - x))$ is the inverse hyperbolic tangent. This $2 \operatorname{arsinh}$ formula can be thought of as coming from the



The distance between two points in the half-plane model can be computed in terms of Euclidean distances in an isosceles trapezoid formed by the points and their reflection across the *x*-axis: a "side length" s, a "diagonal" d, and two "heights" h_1 and h_2. It is the logarithm $\operatorname{dist}(p_1, p_2) = \log\left((s + d)^2 / h_1 h_2\right)$

chord length in the Minkowski metric between points in the hyperboloid model, $\operatorname{chord}(p_1, p_2) = 2\sinh\frac{1}{2}\operatorname{dist}(p_1, p_2)$, analogous to finding arclength on a sphere in terms of chord length. This $2\operatorname{artanh}$ formula can be thought of as coming from Euclidean distance in the Poincaré disk model with one point at the origin, analogous to finding arclength on the sphere by taking a stereographic projection centered on one point and measuring the Euclidean distance in the plane from the origin to the other point.

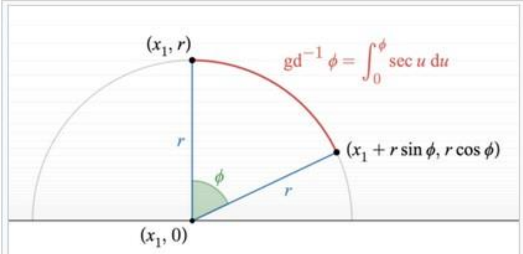


Distance between two points can alternately be computed using ratios of Euclidean distances to the ideal points at the ends of the hyperbolic line.

If the two points p_1 and p_2 are on a hyperbolic line (Euclidean half-circle) which intersects the *x*-axis at the ideal points $p_0 = \langle x_0, 0\rangle$ and $p_3 = \langle x_3, 0\rangle$, the distance from p_1 to p_2 is:

$$\operatorname{dist}(p_1, p_2) = \left|\ln \frac{\|p_2 - p_0\|\|p_1 - p_3\|}{\|p_1 - p_0\|\|p_2 - p_3\|}\right|.$$

Cf. Cross-ratio.



Distance from the apex of a semicircle to another point on it is the inverse Gudermannian function of the central angle.

Some special cases can be simplified. Two points with the same x coordinate:[1]

$$\operatorname{dist}(\langle x, y_1\rangle, \langle x, y_2\rangle) = \left|\ln \frac{y_2}{y_1}\right| = |\ln(y_2) - \ln(y_1)|\,.$$

Two points with the same y coordinate:

$$\operatorname{dist}(\langle x_1, y\rangle, \langle x_2, y\rangle) = 2\operatorname{arsinh} \frac{|x_2 - x_1|}{2y}.$$

One point $\langle x_1, r\rangle$ at the apex of the semicircle $(x - x_1)^2 + y^2 = r^2$, and another point at a central angle of ϕ.

$$\operatorname{dist}(\langle x_1, r\rangle, \langle x_1 \pm r\sin\phi, r\cos\phi\rangle) = 2\operatorname{artanh}\bigl(\tan\tfrac{1}{2}\phi\bigr) = \operatorname{gd}^{-1}\phi,$$

where gd^{-1} is the inverse Gudermannian function, and $\operatorname{artanh} x = \frac{1}{2} \ln \frac{1+x}{1-x}$ is the inverse hyperbolic tangent.

Special points and curves [edit]

- **Ideal points** (points at infinity) in the Poincaré half-plane model are of two kinds:
  - the points on the *x*-axis, and
  - one imaginary point at $y = \infty$ which is the ideal point to which all lines orthogonal to the *x*-axis converge.
- **Straight lines**, geodesics (the shortest path between the points contained within it) are modeled by either:
  - half-circles whose origin is on the x-axis
  - straight vertical rays orthogonal to the x-axis
- A **circle** (curves equidistant from a central point) with center (x, y) and radius r is modeled by:
  a circle with center $(x, y\cosh(r))$ and radius $y\sinh(r)$
- A **hypercycle** (a curve equidistant from a straight line, its axis) is modeled by either:
  - a circular arc which intersects the *x*-axis at the same two ideal points as the half-circle which models its axis but at an acute or obtuse angle
  - a straight line which intersects the *x*-axis at the same point as the vertical line which models its axis, but at an acute or obtuse angle.
- A **horocycle** (a curve whose normals all converge asymptotically in the same direction, its center) is modeled by either:
  - a circle tangent to the *x*-axis (but excluding the ideal point of intersection, which is its center)
  - a line parallel to the *x*-axis, in this case the center is the ideal point at $y = \infty$.

Euclidean synopsis [edit]

A Euclidean circle with center $\langle x_e, y_e \rangle$ and radius r_e represents:

- when the circle is completely inside the halfplane a hyperbolic circle with center
  $\left(x_e, \sqrt{y_e^2 - r_e^2}\right)$

and radius

$$\frac{1}{2}\ln\left(\frac{y_e + r_e}{y_e - r_e}\right).$$

- when the circle is completely inside the halfplane and touches the boundary a horocycle centered around the ideal point $(x_e, 0)$
- when the circle intersects the boundary orthogonal $(y_e = 0)$ a hyperbolic line
- when the circle intersects the boundary non- orthogonal a hypercycle.

Compass and straightedge constructions [edit]

See also: Compass and straightedge constructions

Here is how one can use compass and straightedge constructions in the model to achieve the effect of the basic constructions in the hyperbolic plane.[2] For example, how to construct the half-circle in the Euclidean half-plane which models a line on the hyperbolic plane through two given points.

Creating the line through two existing points [edit]

Draw the line segment between the two points. Construct the perpendicular bisector of the line segment. Find its intersection with the *x*-axis. Draw the circle around the intersection which passes through the given points. Erase the part which is on or below the *x*-axis.

Or in the special case where the two given points lie on a vertical line, draw that vertical line through the two points and erase the part which is on or below the *x*-axis.

Appendix B: Poincare Disk

Poincaré disk model

10 languages

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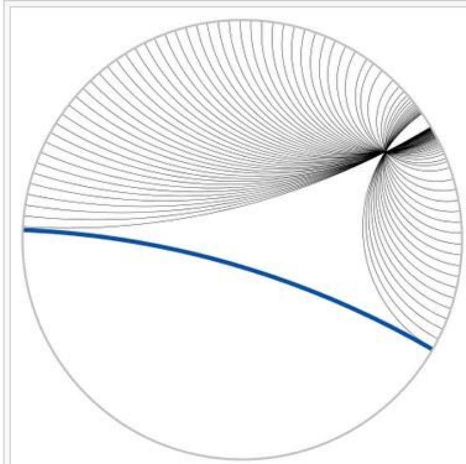
In geometry, the **Poincaré disk model**, also called the **conformal disk model**, is a model of 2-dimensional hyperbolic geometry in which all points are inside the unit disk, and straight lines are either circular arcs contained within the disk that are orthogonal to the unit circle or diameters of the unit circle.

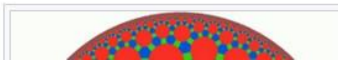


Poincaré disk with hyperbolic parallel lines

The group of orientation preserving isometries of the disk model is given by the projective special unitary group $\mathrm{PSU}(1,1)$, the quotient of the special unitary group SU(1,1) by its center $\{I, -I\}$.

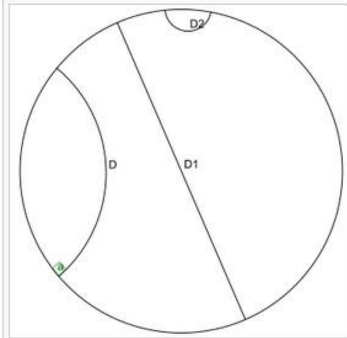
Along with the Klein model and the Poincaré half-space model, it was proposed by Eugenio Beltrami who used these models to show that hyperbolic geometry was equiconsistent with Euclidean geometry. It is named after Henri Poincaré, because his rediscovery of this representation fourteen years later became better known than the original work of Beltrami.[1]

The **Poincaré ball model** is the similar model for *3* or *n*-dimensional hyperbolic geometry in which the points of the geometry are in the *n*-dimensional unit ball.



Lines [edit]

Hyperbolic **straight lines** consist of all arcs of Euclidean circles contained within the disk that are orthogonal to the boundary of the disk, plus all diameters of the disk.



Poincaré disk with 3 ultraparallel (hyperbolic) straight lines

Compass and straightedge construction [edit]

The unique hyperbolic line through two points P and Q not on a diameter of the boundary circle can be constructed by:

- let P' be the inversion in the boundary circle of point P
- let Q' be the inversion in the boundary circle of point Q
- let M be the midpoint of segment PP'
- let N be the midpoint of segment QQ'
- Draw line m through M perpendicular to segment PP'
- Draw line n through N perpendicular to segment QQ'
- let C be where line m and line n intersect.
- Draw circle c with center C and going through P (and Q).
- The part of circle c that is inside the disk is the hyperbolic line.

If P and Q are on a diameter of the boundary circle that diameter is the hyperbolic line.

Another way is:

- let M be the midpoint of segment PQ
- Draw line m through M perpendicular to segment PQ
- let P' be the inversion in the boundary circle of point P
- let N be the midpoint of segment PP'
- Draw line n through N perpendicular to segment PP'
- let C be where line m and line n intersect.
- Draw circle c with center C and going through P (and Q).
- The part of circle c that is inside the disk is the hyperbolic line.

Distance [edit]

Distances in this model are Cayley–Klein metrics. Given two distinct points *p* and *q* inside the disk, the unique hyperbolic line connecting them intersects the boundary at two ideal points, *a* and *b*, label them so that the points are, in order, *a*, *p*, *q*, *b* and $|aq| > |ap|$ and $|pb| > |qb|$.

The hyperbolic distance between *p* and *q* is then

$$d(p,q) = \ln \frac{|aq|\ |pb|}{|ap|\ |qb|}.$$

The vertical bars indicate Euclidean length of the line segment connecting the points between them in the model (not along the circle arc), ln is the natural logarithm.

Another way to calculate the hyperbolic distance between two points is

$$\operatorname{arcosh}\left(1 + \frac{2|pq|^2|r|^2}{(|r|^2 - |op|^2)(|r|^2 - |oq|^2)}\right)$$

where $|op|$ and $|oq|$ are the distances of *p* respective *q* to the centre of the disk, $|pq|$ the distance between *p* and *q*, $|r|$ the radius of the boundary circle of the disk and arcosh is the inverse hyperbolic function of hyperbolic cosine.

When the disk used is the open unit disk and one of the points is the origin and the Euclidean distance between the points is *r* then the hyperbolic distance is:

$$\ln\left(\frac{1+r}{1-r}\right) = 2\operatorname{artanh} r$$

where artanh is the inverse hyperbolic function of the hyperbolic tangent.

When the disk used is the open unit disk and point $x' = (r', \theta)$ lies between the origin and point $x = (r, \theta)$ (i.e. the two points are on the same radius, have the same polar angle and $1 > r > r' > 0$), their hyperbolic distance is

$$\ln\left(\frac{1+r}{1-r} \cdot \frac{1-r'}{1+r'}\right) = 2(\operatorname{artanh} r - \operatorname{artanh} r').$$

This reduces to the previous formula if $r' = 0$.

Circles [edit]

A **circle** (the set of all points in a plane that are at a given distance from a given point, its center) is a circle completely inside the disk not touching or intersecting its boundary. The hyperbolic center of the circle in the model does not in general correspond to the Euclidean center of the circle, but they are on the same radius of the boundary circle.

Hypercycles [edit]

A **hypercycle** (the set of all points in a plane that are on one side and at a given distance from a given line, its axis) is a Euclidean circle arc or chord of the boundary circle that intersects the boundary circle at a positive but non-right angle. Its axis is the hyperbolic line that shares the same two ideal points. This is also known as an equidistant curve.

Horocycles [edit]

A **horocycle** (a curve whose normal or perpendicular geodesics all converge asymptotically in the same direction[*further explanation needed*]), is a circle inside the disk that touches the boundary circle of the disk. The point where it touches the boundary circle is not part of the horocycle. It is an ideal point and is the hyperbolic center of the horocycle.

Euclidean synopsis [edit]

A Euclidean circle:

- that is completely inside the disk is a **hyperbolic circle**.
  (When the center of the disk is not inside the circle, the Euclidean center is always closer to the center of the disk than what the hyperbolic center is, i.e. $t_e < t_h$ holds.)
- that is inside the disk and touches the boundary is a **horocycle**;
- that intersects the boundary orthogonally is a **hyperbolic line**; and
- that intersects the boundary non-orthogonally is a **hypercycle**.

A Euclidean chord of the boundary circle:

- that goes through the center is a hyperbolic line; and
- that does not go through the center is a hypercycle.

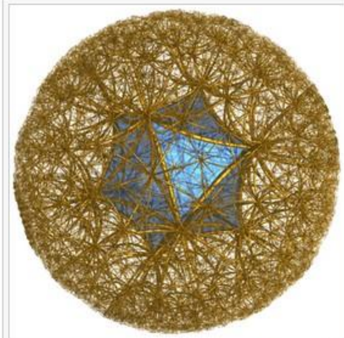
Metric and curvature [edit]

If u and v are two vectors in real n-dimensional vector space $\mathbf{R}^n$ with the usual Euclidean norm, both of which have norm less than 1, then we may define an isometric invariant by

$$\delta(u,v) = 2\frac{\|u-v\|^2}{(1-\|u\|^2)(1-\|v\|^2)},$$

where $\|\cdot\|$ denotes the usual Euclidean norm. Then the distance function is

$$\begin{aligned} d(u,v) &= \operatorname{arcosh}(1+\delta(u,v)) \\ &= 2\operatorname{arsinh}\sqrt{\frac{\delta(u,v)}{2}} \\ &= 2\ln\frac{\|u-v\|+\sqrt{\|u\|^2\|v\|^2-2u\cdot v+1}}{\sqrt{(1-\|u\|^2)(1-\|v\|^2)}}. \end{aligned}$$



Poincaré 'ball' model view of the hyperbolic regular icosahedral honeycomb, {3,5,3}

Such a distance function is defined for any two vectors of norm less than one, and makes the set of such vectors into a metric space which is a model of hyperbolic space of constant curvature −1. The model has the conformal property that the angle between two intersecting curves in hyperbolic space is the same as the angle in the model.

The associated metric tensor of the Poincaré disk model is given by[6]

$$ds^2 = 4\frac{\sum_i dx_i^2}{\left(1-\sum_i x_i^2\right)^2} = \frac{4\|d\mathbf{x}\|^2}{\left(1-\|\mathbf{x}\|^2\right)^2}$$

where the x_i are the Cartesian coordinates of the ambient Euclidean space. The geodesics of the disk model are circles perpendicular to the boundary sphere S^{n-1}.

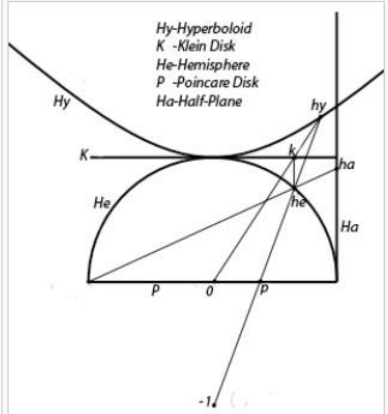
An orthonormal frame with respect to this Riemannian metric is given by

$$e_i = \frac{1}{2}\left(1-|\mathbf{x}|^2\right)\frac{\partial}{\partial x^i},$$

with dual coframe of 1-forms

$$\theta^i = \frac{2}{1-|\mathbf{x}|^2}\,dx^i.$$

Relation to the Klein disk model [edit]



the Poincaré disk model (line *P*), and their relations with the other models

The Klein disk model (also known as the Beltrami–Klein model) and the Poincaré disk model are both models that project the whole hyperbolic plane in a disk. The two models are related through a projection on or from the hemisphere model. The Klein disk model is an orthographic projection to the hemisphere model while the Poincaré disk model is a stereographic projection.

An advantage of the Klein disk model is that lines in this model are Euclidean straight chords. A disadvantage is that the Klein disk model is not conformal (circles and angles are distorted).

When projecting the same lines in both models on one disk both lines go through the same two ideal points. (the ideal points remain on the same spot) also the pole of the chord in the Klein disk model is the center of the circle that contains the arc in the Poincaré disk model.

A point (*x*,*y*) in the Poincaré disk model maps to $\left(\frac{2x}{1+x^2+y^2}\ ,\ \frac{2y}{1+x^2+y^2}\right)$ in the Klein model.

A point (*x*,*y*) in the Klein model maps to $\left(\frac{x}{1+\sqrt{1-x^2-y^2}}\ ,\ \frac{y}{1+\sqrt{1-x^2-y^2}}\right)$ in the Poincaré disk model.

For ideal points $x^2+y^2=1$ and the formulas become $x=x\ ,\ y=y$ so the points are fixed.

If u is a vector of norm less than one representing a point of the Poincaré disk model, then the corresponding point of the Klein disk model is given by:

$$s=\frac{2u}{1+u\cdot u}.$$

Conversely, from a vector s of norm less than one representing a point of the Beltrami–Klein model, the corresponding point of the Poincaré disk model is given by:

$$u=\frac{s}{1+\sqrt{1-s\cdot s}}=\frac{\left(1-\sqrt{1-s\cdot s}\right)s}{s\cdot s}.$$

Relation to the Poincaré half-plane model [edit]

See also: Cayley transform § Complex homography

The Poincaré disk model and the Poincaré half-plane model are both named after Henri Poincaré.

If u is a complex number of norm less than one representing a point of the Poincaré disk model, then the corresponding point of the half-plane model is given by the inverse of the Cayley transform:

$$s=\frac{u+i}{iu+1}.$$

A point (*x*,*y*) in the disk model maps to $\left(\frac{2x}{x^2+(1-y)^2}\ ,\ \frac{1-x^2-y^2}{x^2+(1-y)^2}\right)$ in the halfplane model.[7]

A point (*x*,*y*) in the halfplane model maps to $\left(\frac{2x}{x^2+(1+y)^2}\ ,\ \frac{x^2+y^2-1}{x^2+(1+y)^2}\right)$ in the disk model.

Appendix B: Beltrami-Klein Disk

Beltrami–Klein model

8 languages

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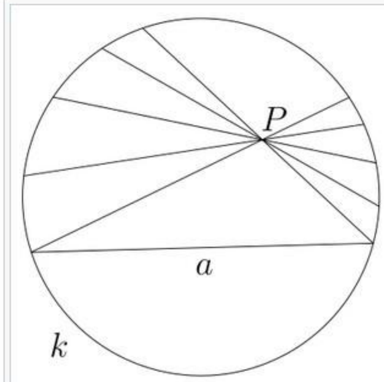
In geometry, the **Beltrami–Klein model**, also called the **projective model**, **Klein disk model**, and the **Cayley–Klein model**, is a model of hyperbolic geometry in which points are represented by the points in the interior of the unit disk (or *n*-dimensional unit ball) and lines are represented by the chords, straight line segments with ideal endpoints on the boundary sphere.



Many hyperbolic lines through point P not intersecting line a in the Beltrami Klein model

The **Beltrami–Klein model** is named after the Italian geometer Eugenio Beltrami and the German Felix Klein while "Cayley" in **Cayley–Klein model** refers to the English geometer Arthur Cayley.

The Beltrami–Klein model is analogous to the gnomonic projection of spherical geometry, in that geodesics (great circles in spherical geometry) are mapped to straight lines.

This model is not conformal, meaning that angles and circles are distorted, whereas the Poincaré disk model preserves these.

In this model, lines and segments are straight Euclidean segments, whereas in the Poincaré disk model, lines are arcs that meet the boundary orthogonally.

Distance formula [edit]

The distance function for the Beltrami–Klein model is a Cayley–Klein metric. Given two distinct points *p* and *q* in the open unit ball, the unique straight line connecting them intersects the boundary at two ideal points, *a* and *b*, label them so that the points are, in order, *a*, *p*, *q*, *b* and $|aq| > |ap|$ and $|pb| > |qb|$.

The hyperbolic distance between *p* and *q* is then: $d(p,q) = \frac{1}{2} \ln \frac{|aq|\ |pb|}{|ap|\ |qb|}$

The vertical bars indicate Euclidean distances between the points in the model, ln is the natural logarithm and the factor of one half is needed to give the model the standard curvature of −1.

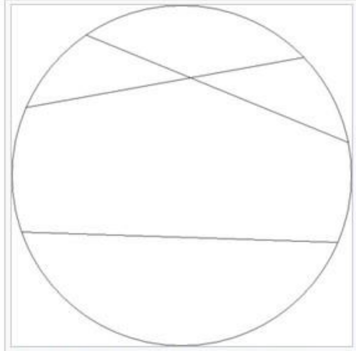
When one of the points is the origin and Euclidean distance between the points is *r* then the hyperbolic distance is:

$$\frac{1}{2} \ln \left(\frac{1+r}{1-r} \right) = \operatorname{artanh} r,$$

where *artanh* is the inverse hyperbolic function of the hyperbolic tangent.

The Klein disk model [edit]

In two dimensions the **Beltrami–Klein model** is called the **Klein disk model**. It is a disk and the inside of the disk is a model of the entire hyperbolic plane. Lines in this model are represented by chords of the boundary circle (also called the *absolute*). The points on the boundary circle are called **ideal points**; although well defined, they do not belong to the hyperbolic plane. Neither do points outside the disk, which are sometimes called **ultra ideal points**.



Lines in the projective model of the hyperbolic plane

The model is not conformal, meaning that angles are distorted, and circles on the hyperbolic plane are in general not circular in the model. Only circles that have their centre at the centre of the boundary circle are not distorted. All other circles are distorted, as are horocycles and hypercycles

Properties [edit]