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CHAPTER 5

A Locally Well-Behaved Potential Function and a Simple Newton-Type Method for Finding the Center of a Polytope

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Abstract. The center of a bounded full-dimensional polytope $P = \{x: Ax \geq b\}$ is the unique point ω that maximizes the strictly concave potential function $F(x) = \sum_{i=1}^{m} \ln(a_i^T x - b_i)$ over the interior of P. Let x_0 be a point in the interior of P. We show that the first two terms in the power series of $F(x)$ at x_0 serve as a good approximation to $F(x)$ in a suitable ellipsoid around x_0 and that minimizing the first-order (linear) term in the power series over this ellipsoid increases $F(x)$ by a fixed additive constant as long as x_0 is not too close to the center ω.

§1. Introduction

Let the polytope P be defined as

$$P = \{x: Ax \geq b\}$$

where $x \in R^n$, $b \in R^m$, and $A \in R^{m \times n}$. The center ω of the polytope P is defined to be the unique point that maximizes the strictly concave potential function

$$F(x) = \sum_{i=1}^{m} \ln(a_i^T x - b_i).$$

Under the assumption that the polytope P is bounded and has a nonzero interior, the Hessian of F is negative definite over the interior of P, and so ω is indeed a unique point. We let $f(x) = F(\omega) - F(x)$ denote the normalized potential corresponding to $F(x)$. We define transformed coordinates $\Psi_i(x)$ as

$$\Psi_i(x) = \frac{a_i^T(x - \omega)}{a_i^T \omega - b_i}, \qquad i = 1, 2, \ldots, m.$$

The coordinates $\Psi_i(x)$ were originally defined in [3, 4] and will be used extensively here in proving various properties of $f(x)$. Let $\Sigma(\delta)$ denote the ellipsoid around ω given by

$$\Sigma(\delta) = \left\{x: \sum_{i=1}^{m} \Psi_i(x)^2 \le \delta^2\right\}.$$

Then $\Sigma(1) \subseteq P \subseteq \Sigma(m)$ [3, 4]. Thus the ratio of the maximum to the minimum distance from the center ω to any point on the boundary of P is upper bounded by m. So the center is a balanced point in the polytope. The center plays an important role in algorithms for linear and convex programming [1–6]. In particular, a polynomial time algorithm for computing a good approximation to the center can be converted into a polynomial time algorithm for linear programming [3, 6].

In this short chapter we shall describe some properties of $f(x)$ and see how they may be used to develop an algorithm for finding the center ω. We shall assume that an initial point strictly in the interior of the polytope is available.

§2. Local Behavior of the Potential

Let x_0 be a point in the interior of the polytope P, let η be the gradient of $f(x)$ evaluated at x_0, and let H denote the Hessian of $f(x)$ evaluated at x_0. Explicitly,

$$\eta = -\sum_{i=1}^{m} \frac{1}{(a_i^T x_0 - b_i)} a_i$$

and

$$H = \sum_{i=1}^{m} \frac{1}{(a_i^T x_0 - b_i)^2} a_i a_i^T.$$

Let $E(r)$ be the ellipsoid around x_0 defined as

$$E(r) = \{x: (x - x_0)^T H (x - x_0) \le r^2\}.$$

Note that if $0 \le r \le 1$ then the ellipsoid $E(r)$ is contained within the polytope P. We shall show that minimizing the linear function $\eta^T x$ over the ellipsoid $E(r)$ gives a good reduction in $f(x)$. Specifically, $f(x)$ is reduced by at least a fixed additive constant if x_0 lies outside the ellipsoid $\Sigma(0.5)$, whereas $f(x)$ is reduced by at least a fixed fraction if x_0 is within the ellipsoid $\Sigma(0.5)$.

Let $x = x_0 + t\xi$. Using the power series expansion of $f(x_0 + t\xi)$ at x_0, $f(x_0 + t\xi)$ may be written as

$$f(x_0 + t\xi) = f(x_0) + t\eta^T \xi + \frac{t^2}{2} \xi^T H \xi + \sum_{j=3}^{\infty} \frac{(-1)^j t^j}{j} \left(\sum_{i=1}^{m} \frac{(a_i^T \xi)^j}{(a_i^T x_0 - b_i)^j} \right).$$

We shall prove the following lemmas.

Lemma 5.1. *Let r be a parameter such that $0 \le r < 1$, and let $x = x_0 + t\xi$ be a point in the ellipsoid $E(r)$ around x_0. Then*

$$\frac{t^2}{2}\xi^T H\xi \le 0.5r^2$$

and

$$\left|\sum_{j=3}^{\infty}\frac{(-1)^j t^j}{j}\left(\sum_{i=1}^{m}\frac{(a_i^T\xi)^j}{(a_i^T x_0 - b_i)^j}\right)\right| \le \frac{r^3}{3(1-r)}.$$

The next lemma lower bounds the maximum change obtainable in the linear function $\eta^T x$ over the ellipsoid $E(r)$.

Lemma 5.2. *Let r be a parameter such that $0 \le r < 1$, and let δ be a parameter such that $0 \le \delta < 1$. Let x' be the point where the straight line joining x_0 to the center ω intersects the boundary of the ellipsoid $E(r)$ around x_0. The point x' satisfies the following conditions.*

(1) *If $x_0 \notin \Sigma(\delta)$ then* $\eta^T(x_0 - x') \ge \dfrac{\delta(1-\delta)}{2(1+\delta)}r.$

(2) *If $x_0 \in \Sigma(\delta)$ then* $\eta^T(x_0 - x') \ge ((1-\delta)f(x_0))^{1/2}r.$

The final lemma says that the point that minimizes the linear function $\eta^T x$ over $E(r)$ gives a good reduction in $f(x)$.

Lemma 5.3. *Let δ be a parameter such that $0 < \delta < 0.7$, and let ε be a parameter such that $0 < \varepsilon < 1$. Let r_0 be defined by*

$$r_0 = \begin{cases}\varepsilon, & \text{if } x_0 \notin \Sigma(\delta)\\ \varepsilon\sqrt{f(x_0)}, & \text{if } x_0 \in \Sigma(\delta)\end{cases}.$$

Let $\mathbf{x}$ *be the point that minimizes the linear function $\eta^T x$ over the ellipsoid $E(r_0)$ around x_0. Then point* $\mathbf{x}$ *satisfies the following conditions.*

(1) *If $x_0 \notin \Sigma(\delta)$ then*

$$f(\mathbf{x}) - f(x_0) \le -\frac{\delta(1-\delta)}{2(1+\delta)}\varepsilon + 0.5\varepsilon^2 + \frac{\varepsilon^3}{3(1-\varepsilon)}$$

(2) *If $x_0 \in \Sigma(\delta)$ then*

$$f(\mathbf{x}) \le \left(1 - \varepsilon(1-\delta)^{1/2} + 0.5\varepsilon^2 + \frac{\varepsilon^3}{3(1-\varepsilon)}\right)f(x_0).$$

Lemma 5.3 may be converted into an algorithm for finding the center ω as follows. Let x_0 be the current point and let $\mathbf{x}$ be the point that minimizes $\eta^T x$ over the ellipsoid $E(r_0)$, where r_0 is as defined in Lemma 5.3. $x_0 - \mathbf{x}$ satisfies the system of linear equations

$$H(x_0 - \mathbf{x}) = \mathbf{t}\eta$$

for some scalar $\mathbf{t}$. So we may compute a direction $\boldsymbol{\xi}$ by solving the system

$$H\xi = \eta$$

and minimize $f(x)$ on the line $x_0 + t\xi$. (Note that the one-dimensional minimization need not be exact.) According to Lemma 5.3, the point thus obtained will reduce $f(x)$ by an additive constant if $x_0 \notin \Sigma(0.5)$ and will reduce $f(x)$ by a fixed fraction if $x_0 \in \Sigma(0.5)$. Thus starting with an initial point x_{init} strictly in the interior of the polytope P we can produce a sequence of points converging to the center ω.

We now give a more formal algorithm for computing the center. We shall assume that a point x_{init} such that $f(x_{\text{init}}) \leq M$ is available. (x_{init} is in the interior of polytope P.) The algorithm produces a sequence of points $z_0 = x_{\text{init}}$, $z_1, \ldots, z_k, \ldots$ that converge to the center ω. x_0 denotes the current point in the computation, and k is the step number. Let θ be a parameter less than 1/250. The output of the the algorithm is a point z_q such that $f(z_q) \leq \theta$.

Algorithm Find-Center
$x_0 := x_{\text{init}}$; $z_0 := x_{\text{init}}$; $k := 0$;
Loop:
  /*x_0 **is the current point** */
  Let η be the gradient of $f(x)$ at x_0 and
  let H be the Hessian of $f(x)$ at x_0;
  Let ξ be the direction obtained by solving $H\xi = \eta$;
  /* **Increment step number** k **and compute** z_k */
  $k := k + 1$;
  Let z_k be the point that minimizes $f(x)$ on the line
  $x_0 + t\xi$ where t is a scalar;
  /* **Reset** x_0 */
  $x_0 := z_k$;
  If $f(z_k - 1) - f(z_k) \geq \theta/3$ then go to Loop
  else halt;
end Find-Center

It is worth noting that ξ is in the same direction as the direction generated by the Newton-Raphson method applied to the problem of minimizing $f(x)$; however, the length of the step taken in the direction of ξ is quite different in the above algorithm. Also note that in the above algorithm the value of F at ω is not required to compute the difference $f(z_{k-1}) - f(z_k)$. Furthermore, the one- dimensional minimization on the line $x_0 + t\xi$ need not be performed exactly. (It suffices to find a point on this line where f is either reduced by a fixed additive constant or f is reduced by a fixed fraction.)

Let z_q be the point at the termination of the algorithm. Then $f(z_{q-2}) \geq \theta/3$, and $f(z_q) \leq \theta$. That $f(z_{q-2}) \geq \theta/3$ follows from the observations that the algorithm did not terminate at the $(q-1)$st step and that the minimum value of f is zero. That $f(z_q) \leq \theta$ is shown as follows. We have that

$$f(z_{q-1}) - f(z_q) < \theta/3,$$

and hence by Lemma 5.3 (with $\delta = 1/2$, $\varepsilon = 1/10$), z_{q-1} cannot be outside

$\Sigma(0.5)$. Therefore $z_{q-1} \in \Sigma(0.5)$. Then again by Lemma 5.3 (with $\delta = 1/2$, $\varepsilon = 0.5$) we get that

$$f(z_{q-1}) - f(z_q) \geq \tfrac{1}{3} f(z_{q-1}).$$

Thus $f(z_q) \leq f(z_{q-1}) \leq \theta$.

We shall now show an upper bound of $O((mn^2 + n^3)(M + \log(1/\theta)))$ on the total number of arithmetic operations performed by the algorithm. Since during a step we must compute the gradient η and the Hessian H and solve a system of linear equations, executing a step requires $O(mn^2 + n^3)$ arithmetic operations. So to obtain the said bound on the total number of arithmetic operations it suffices to show a bound of $O(M + \log(1/\theta))$ on the total number of steps executed by the algorithm. The number of steps is upper bounded as follows. It requires $O(M)$ steps to obtain a point in $\Sigma(0.3)$ because by Lemma 5.3 $f(x)$ is decreased by a fixed additive constant at each step as long as the current point is outside $\Sigma(0.3)$. Let p be the step number such that $z_p \in \Sigma(0.3)$ and for $0 \leq k < p$, $z_k \notin \Sigma(0.3)$. By Lemma 5.5 in Section 3, we get that $f(z_p) \leq 0.1$ and that for all $k > p$, $f(z_k) \leq f(z_p) \leq 0.1$. Then by Lemma 5.6 in Section 3 it follows that $z_k \in \Sigma(0.5)$ for all $k > p$. So we can apply Lemma 5.3 to each step after the pth step and conclude that from the pth step onward $f(x)$ decreases by a fixed fraction at each step. Hence in $O(\log(1/\theta))$ steps after the pth step $f(x)$ must fall below $\theta/3$. Thus the total number of steps is $O(M + \log(1/\theta))$.

In a manner similar to [2, 6] it is possible to reduce the time complexity of the above algorithm by using an approximate Hessian H_a at each step where

$$H_a = \sum_{i=1}^{m} \frac{\Delta_i}{(a_i^T x_0 - b_i)^2} a_i a_i^T$$

and $\Delta_i \in [1/1.1, 1.1]$, $1 \leq i \leq m$. The above algorithm is modified as follows. The direction ξ is now computed by solving the system $H_a \xi = \eta$ instead of the system $H\xi = \eta$. The approximate Hessian H_a and its inverse are maintainedby performing rank-one updates as described in [2, 6]. z_k is still obtained by a one-dimensional minimization on the line $x_0 + t\xi$. We note that the direction ξ computed by the modified algorithm is now quite different from the Newton-Raphson direction. The modified algorithm still requires $O(M + \log(1/\theta))$ steps, which may be seen as follows. Suppose $E_a(r)$ is the ellipsoid defined as

$$E_a(r) = \{x: (x - x_0)^T H_a (x - x_0) \leq r^2\}.$$

Then

$$E\left(\frac{r}{1.1}\right) \subseteq E_a(r) \subseteq E(1.1r).$$

So Lemmas 5.1 through 5.3 are still valid (but with different constants) if the ellipsoid $E(r)$ is replaced by the ellipsoid $E_a(r)$. Thus minimizing $\eta^T x$ over $E_a(r)$ rather than $E(r)$ still gives an adequate decrease in $f(x)$. Use of approximate Hessians reduces the average work per step to $O(mn + m^{0.5} n^2)$ arithmetic operations, leading to a total of $O((mn + m^{0.5}n^2)(M + \log(1/\theta)))$ arithmetic operations.

§3. Proofs of Lemmas

We shall first prove a couple of lemmas that will be used in the proofs of the lemmas stated in Section 2.

Lemma 5.4. *Let* $\Psi_i(x) = (a_i^T x - a_i^T \omega)/(a_i^T \omega - b_i)$, *for* $i = 1, 2, \ldots, m$. *Then for any point* x *in the polytope* P,

$$\sum_{i=1}^{m} \Psi_i(x) = 0.$$

PROOF. A proof is given in [3, 4] but we include it here for completeness. Since the gradient of $f(x)$ vanishes at ω, taking the dot product of the gradient of $f(x)$ at ω with $\omega - x$ gives

$$-\sum_{i=1}^{m} \frac{a_i^T(\omega - x)}{(a_i{}^T \omega - b_i)} = \sum_{i=1}^{m} \Psi_i(x) = 0.$$

The next lemma bounds for the maximum value of $f(x)$ in the region $\Sigma(\delta)$ for $0 \le \delta < 1$.

Lemma 5.5. *Let* δ *be a parameter such that* $0 \le \delta < 1$. *Then the maximum value of* $f(x)$ *in the ellipsoid* $\Sigma(\delta)$ *is at most* $\delta^2/2(1 - \delta)$.

PROOF. Since $f(x)$ is strictly convex, and ω minimizes $f(x)$, the maximum value of $f(x)$ over the region $\Sigma(\delta)$ is achieved on the boundary of $\Sigma(\delta)$. We have

$$f(x) = \sum_{i=1}^{m} \ln\left(\frac{1}{1 + \Psi_i(x)}\right).$$

Using the Taylor series expansion, on the boundary of $\Sigma(\delta)$ we may write $f(x)$ as

$$f(x) = \sum_{j=1}^{\infty} \sum_{i=1}^{m} \frac{(-1)^j \Psi_i(x)^j}{j}.$$

From Lemma 5.4, $\sum_{i=1} \Psi_i(x) = 0$. So on the boundary of $\Sigma(\delta)$ we get

$$\begin{aligned} f(x) &= \sum_{j=2}^{\infty} \sum_{i=1}^{m} \frac{(-1)^j \Psi_i(x)^j}{j} \\ &\le \sum_{i=1}^{m} \frac{\Psi_i(x)^2}{2}\left(1 + \frac{2\delta}{3} + \frac{2\delta^2}{4} + \frac{2\delta^3}{5} + \cdots\right) \\ &\le \frac{\delta^2}{2(1 - \delta)}. \end{aligned}$$

The next lemma lower bounds the value of the function $\sum_{i=1}^{m}(1/(1+\Psi_i(x))-1)$ over the region $\{x\colon x \in P, x \notin \Sigma(\delta)\}$ for $0 < \delta < 1$.

Lemma 5.6. *Let δ be a parameter such that $0 < \delta < 1$. The minimum value of $\sum_{i=1}^{m}(1/(1+\Psi_i(x))-1)$ over the region $\{x\colon x \in P, x \notin \Sigma(\delta)\}$ is greater than or equal to $\delta^2/(1+\delta)$.*

PROOF. Let $g(x) = \sum_{i=1}^{m}(1/(1+\Psi_i(x))-1)$. By Lemma 5.4, $\sum_{i=1}^{m}\Psi_i(x) = 0$, and so

$$g(x) = \sum_{i=1}^{m}\left(\frac{1}{1+\Psi_i(x)} + \Psi_i(x) - 1\right) = \sum_{i=1}^{m}\frac{\Psi_i(x)^2}{1+\Psi_i(x)}.$$

The Hessian of $g(x)$ evaluated at a point x is the matrix $A^T D(x) A$ where $D(x)$ is a diagonal matrix whose ith diagonal entry $D_{ii}(x)$ is given by $D_{ii}(x) = 2(a_i^T\omega - b_i)/(a_i^T x - b_i)^3$. As $A^T D(x) A$ is positive definite in the interior of P, $g(x)$ is strictly convex in the interior of P. Furthermore, the minimum value of $g(x)$ over the interior of P occurs at the center ω. Thus the minimum value of $g(x)$ over the region $\{x\colon x \in P, x \notin \Sigma(\delta)\}$ occurs on the boundary of $\Sigma(\delta)$. And on the boundary of $\Sigma(\delta)$,

$$g(x) = \sum_{i=1}^{m}\frac{\Psi_i(x)^2}{1+\Psi_i(x)} \geq \sum_{i=1}^{m}\frac{\Psi_i(x)^2}{1+\delta} \geq \frac{\delta^2}{1+\delta}.$$

We shall now prove the lemmas stated in Section 2. We shall restate the lemmas for convenience.

Lemma 5.1. *Let r be a parameter such that $0 \leq r < 1$, and let $x = x_0 + t\xi$ be a point in the ellipsoid $E(r)$ around x_0. Then*

$$\frac{t^2}{2}\xi^T H \xi \leq 0.5 r^2$$

and

$$\left|\sum_{j=3}^{\infty}\frac{(-1)^j t^j}{j}\left(\sum_{i=1}^{m}\frac{(a_i^T\xi)^j}{(a_i^T x_0 - b_i)^j}\right)\right| \leq \frac{r^3}{3(1-r)}.$$

PROOF. By definition of $E(r)$,

$$t^2\xi^T H\xi \leq r^2.$$

As H may be written as

$$H = \sum_{i=1}^{m}\frac{1}{(a_i^T x_0 - b_i)^2}a_i a_i^T$$

we get

$$t^2\xi^T H\xi = t^2\sum_{i=1}^{m}\frac{(a_i^T\xi)^2}{(a_i^T x_0 - b_i)^2} \leq r^2.$$

Thus

$$\left| \sum_{j=3}^{\infty} \frac{(-1)^j t^j}{j} \left(\sum_{i=1}^{m} \frac{(a_i^T \xi)^j}{(a_i^T x_0 - b_i)^j} \right) \right| \le \sum_{j=3}^{\infty} \frac{r^j}{j}$$

$$\le \frac{r^3}{3(1-r)}.$$

Lemma 5.2. *Let r be a parameter such that $0 \le r < 1$, and let δ be a parameter such that $0 \le \delta < 1$. Let x' be the point where the straight line joining x_0 to the center ω intersects the boundary of the ellipsoid $E(r)$* around x_0. *The point x' satisfies the following conditions.*

(1) *If $x_0 \notin \Sigma(\delta)$ then* $\eta^T(x_0 - x') \ge \dfrac{\delta(1-\delta)}{2(1+\delta)} r.$

(2) *If $x_0 \in \Sigma(\delta)$ then* $\eta^T(x_0 - x') \ge ((1-\delta) f(x_0))^{1/2} r.$

Proof. Let $x_0 - x' = \lambda u$ where u is the unit vector in the direction of $x_0 - x'$ and $\lambda = \|x_0 - x'\|_2$. Then

$$\lambda^2 u^T H u \ge r^2$$

and so

$$\lambda \ge \frac{r}{\sqrt{u^T H u}}.$$

Thus

$$\eta^T(x_0 - x') \ge \frac{\eta^T u}{\sqrt{u^T H u}} r$$

$$\ge \frac{\eta^T(x_0 - \omega)}{\sqrt{(x_0 - \omega)^T H (x_0 - \omega)}} r \tag{5.1}$$

Note that

$$\psi_i(x_0) = \frac{a_i^T x_0 - a_i^T \omega}{(a_i^T \omega - b_i)}, \qquad i = 1, 2, \ldots, m,$$

$$\eta = -\sum_{i=1}^{m} \frac{1}{(a_i^T x_0 - b_i)} a_i,$$

and

$$H = \sum_{i=1}^{m} \frac{1}{(a_i^T x_0 - b_i)^2} a_i a_i^T.$$

Then

$$\eta^T(x_0 - \omega) = \sum_{i=1}^{m} \left(\frac{1}{1 + \psi_i(x_0)} - 1 \right)$$

and

$$(x_0 - \omega)^T H(x_0 - \omega) = \sum_{i=1}^{m} \left(\frac{1}{1 + \psi_i(x_0)} - 1 \right)^2.$$

So from (5.1) we get

$$\eta^T(x_0 - x') \geq \frac{\sum_{i=1}^{m} \left(\frac{1}{1 + \psi_i(x_0)} - 1 \right)}{\sqrt{\sum_{i=1}^{m} \left(\frac{1}{1 + \psi_i(x_0)} - 1 \right)^2}} r \tag{5.2}$$

Also,

$$f(x_0) = \sum_{i=1}^{m} \ln \left(\frac{1}{1 + \psi_i(x_0)} \right) \tag{5.3}$$

From Lemma 5.4,

$$\sum_{i=1}^{m} \psi_i(x_0) = 0.$$

So

$$\sum_{i=1}^{m} \left(\frac{1}{1 + \psi_i(x_0)} - 1 \right) = \sum_{i=1}^{m} \left(\frac{1}{1 + \psi_i(x_0)} + \psi_i(x_0) - 1 \right) = \sum_{i=1}^{m} \frac{\psi_i(x_0)^2}{1 + \psi_i(x_0)} \tag{5.4}$$

Also,

$$\sum_{i=1}^{m} \left(\frac{1}{1 + \psi_i(x_0)} - 1 \right)^2 = \sum_{i=1}^{m} \frac{\psi_i(x_0)^2}{(1 + \psi_i(x_0))^2}.$$

Thus from (5.2) we may conclude that

$$\eta^T(x_0 - x') \geq \frac{\sum_{i=1}^{m} \frac{\psi_i(x_0)^2}{1 + \psi_i(x_0)}}{\left(\sum_{i=1}^{m} \frac{\psi_i(x_0)^2}{(1 + \psi_i(x_0))^2} \right)^{1/2}} r \tag{5.5}$$

From (5.3) and (5.5) it follows that in order to prove the lemma it suffices to show that

(1) If $x_0 \notin \Sigma(\delta)$ then

$$\sum_{i=1}^{m} \frac{\psi_i(x_0)^2}{1 + \psi_i(x_0)} \geq \frac{\delta(1 - \delta)}{2(1 + \delta)} \left(\sum_{i=1}^{m} \frac{\psi_i(x_0)^2}{(1 + \psi_i(x_0))^2} \right)^{1/2}.$$

(2) If $x_0 \in \Sigma(\delta)$ then

$$\sum_{i=1}^{m} \frac{\psi_i(x_0)^2}{1 + \psi_i(x_0)} \geq \left((1 - \delta) \left(\sum_{i=1}^{m} \ln \left(\frac{1}{1 + \psi_i(x_0)} \right) \right) \left(\sum_{i=1}^{m} \frac{\psi_i(x_0)^2}{(1 + \psi_i(x_0))^2} \right) \right)^{1/2}.$$

Case 1. $x \notin \Sigma(\delta)$.

There are two subcases depending on the value of $\sum_{i=1}^{m} [\psi_i(x_0)^2/(1 + \psi_i(x_0))^2]$.

Case 1.1. $\sum_{i=1}^{m} \frac{\psi_i(x_0)^2}{(1 + \psi_i(x_0))^2} \leq \delta^2.$

Since $x_0 \notin \Sigma(\delta)$, from (5.4) and Lemma 5.6 it follows that

$$\sum_{i=1}^{m} \frac{\psi_i(x_0)^2}{1 + \psi_i(x_0)} \geq \frac{\delta^2}{1 + \delta}$$

$$\geq \frac{\delta}{1 + \delta}\left(\sum_{i=1}^{m} \frac{\psi_i(x_0)^2}{(1 + \psi_i(x_0))^2}\right)^{1/2} \tag{5.6}$$

Case 1.2. $\sum_{i=1}^{m} \frac{\psi_i(x_0)^2}{(1 + \psi_i(x_0))^2} \geq \delta^2.$

Note that $1 + \psi_i(x) > 0$ for all points x in the interior of the polytope P. Thus

$$\sum_{|\psi_i(x_0)| \geq \delta} \frac{\psi_i(x_0)^2}{1 + \psi_i(x_0)} \geq \sum_{|\psi_i(x_0)| \geq \delta} \delta \frac{|\psi_i(x_0)|}{|1 + \psi_i(x_0)|}$$

$$\geq \delta\left(\sum_{|\psi_i(x_0)| \geq \delta} \frac{\psi_i(x_0)^2}{(1 + \psi_i(x_0))^2}\right)^{1/2}.$$

Suppose that

$$\sum_{|\psi_i(x_0)| \geq \delta} \frac{\psi_i(x_0)^2}{(1 + \psi_i(x_0))^2} \geq \sum_{|\psi_i(x_0)| < \delta} \frac{\psi_i(x_0)^2}{(1 + \psi_i(x_0))^2}.$$

Then it follows that

$$\sum_{|\psi_i(x_0)| \geq \delta} \frac{\psi_i(x_0)^2}{1 + \psi_i(x_0)} \geq \frac{\delta}{\sqrt{2}}\left(\sum_{i=1}^{m} \frac{\psi_i(x_0)^2}{(1 + \psi_i(x_0))^2}\right)^{1/2} \tag{5.7}$$

So let us assume that

$$\sum_{|\psi_i(x_0)| < \delta} \frac{\psi_i(x_0)^2}{(1 + \psi_i(x_0))^2} \geq \sum_{|\psi_i(x_0)| \geq \delta} \frac{\psi_i(x_0)^2}{(1 + \psi_i(x_0))^2}$$

Then

$$\sum_{|\psi_i(x_0)| < \delta} \frac{\psi_i(x_0)^2}{(1 + \psi_i(x_0))^2} \geq \frac{\delta^2}{2}.$$

Thus

$$\sum_{|\psi_i(x_0)| < \delta} \frac{\psi_i(x_0)^2}{1 + \psi_i(x_0)} \geq (1 - \delta) \sum_{|\psi_i(x_0)| < \delta} \frac{\psi_i(x_0)^2}{(1 + \psi_i(x_0))^2}$$

$$\geq \frac{1}{\sqrt{2}}\delta(1 - \delta)\left(\sum_{|\psi_i(x_0)| < \delta} \frac{\psi_i(x_0)^2}{(1 + \psi_i(x_0))^2}\right)^{1/2}$$

$$\geq \frac{1}{2}\delta(1 - \delta)\left(\sum_{i=1}^{m} \frac{\psi_i(x_0)^2}{(1 + \psi_i(x_0))^2}\right)^{1/2} \tag{5.8}$$

Thus from (5.6), (5.7), and (5.8) we may conclude that for Case 1

$$\sum_{i=1}^{m} \frac{\psi_i(x_0)^2}{1+\psi_i(x_0)} \geq \frac{\delta(1-\delta)}{2(1+\delta)}\left(\sum_{i=1}^{m} \frac{\psi_i(x_0)^2}{(1+\psi_i(x_0))^2}\right)^{1/2}.$$

Case 2. $x_0 \in \Sigma(\delta)$.

Note that since $x_0 \in \Sigma(\delta)$, $|\psi_i(x_0)| \leq \delta$, $i = 1, 2, \ldots, m$. So

$$\sum_{i=1}^{m} \frac{\psi_i(x_0)^2}{1+\psi_i(x_0)} \geq (1-\delta) \sum_{i=1}^{m} \frac{\psi_i(x_0)^2}{(1+\psi_i(x_0))^2}.$$

Also,

$$\sum_{i=1}^{m} \ln\left(\frac{1}{1+\psi_i(x_0)}\right) \leq \sum_{i=1}^{m}\left(\frac{1}{1+\psi_i(x_0)} - 1\right)$$

$$\leq \sum_{i=1}^{m} \frac{\psi_i(x_0)^2}{1+\psi_i(x_0)} \quad \text{(by (5.4))}$$

Thus for Case 2

$$\sum_{i=1}^{m} \frac{\psi_i(x_0)^2}{1+\psi_i(x_0)} \geq \left((1-\delta)\left(\sum_{i=1}^{m} \ln\left(\frac{1}{1+\psi_i(x_0)}\right)\right)\left(\sum_{i=1}^{m} \frac{\psi_i(x_0)^2}{(1+\psi_i(x_0))^2}\right)\right)^{1/2}$$

Lemma 5.3. *Let δ be a parameter such that $0 < \delta < 0.7$, and let ε be a parameter such that $0 < \varepsilon < 1$. Let r_0 be defined by*

$$r_0 = \begin{cases} \varepsilon, & \text{if } x_0 \notin \Sigma(\delta), \\ \varepsilon\sqrt{f(x_0)}, & \text{if } x_0 \in \Sigma(\delta). \end{cases}$$

Let $\hat{x}$ be the point that minimizes the linear function $\eta^T x$ over the ellipsoid $E(r_0)$ around x_0. The point $\hat{x}$ satisfies the following conditions.

(1) *If $x_0 \notin \Sigma(\delta)$ then*

$$f(\hat{x}) - f(x_0) \leq -\frac{\delta(1-\delta)}{2(1+\delta)}\varepsilon + 0.5\varepsilon^2 + \frac{\varepsilon^3}{3(1-\varepsilon)}.$$

(2) *If $x_0 \in \Sigma(\delta)$ then*

$$f(\hat{x}) \leq \left(1 - \varepsilon(1-\delta)^{1/2} + 0.5\varepsilon^2 + \frac{\varepsilon^3}{3(1-\varepsilon)}\right) f(x_0).$$

Proof. Let x' be the point where the straight line joining x_0 and the center ω intersects the boundary of $E(r)$.

Case 1. $x_0 \notin \Sigma(\delta)$.

Proof by application of Lemmas 5.1 and 5.2 above and from the observation that $\eta^T\hat{x} \leq \eta^T x'$.

Case 2. $x_0 \in \Sigma(\delta)$.

Before we may apply Lemma 5.1 we must show that r_0 is less than 1. To show that $r_0 < 1$ it is adequate to prove that $f(x_0) \leq 1$. From Lemma 5.5, $f(x_0) \leq \delta^2/2(1-\delta)$, and since $\delta \leq 0.7$ we get that $f(x_0) \leq 1$. We can now apply Lemmas 5.1 and 5.2, and noting that $\eta^T \hat{x} \leq \eta^T x'$ we may conclude that

$$\begin{aligned} f(\hat{x}) &\leq f(x_0) - r_0\sqrt{(1-\delta)f(x_0)} + 0.5r_0^2 + \frac{r_0^3}{3(1-r_0)} \\ &\leq \left(1 - \varepsilon\sqrt{(1-\delta)} + 0.5\varepsilon^2 + \frac{\varepsilon^3}{3(1-\varepsilon)}\right) f(x_0), \quad \text{as } f(x_0) < 1. \end{aligned}$$

That concludes the proof for Case 2.

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