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The eight-tetrahedra longest-edge partition and Kuhn triangulations

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Abstract

The Kuhn triangulation of a cube is obtained by subdividing the cube into six right-type tetrahedra once a couple of opposite vertices have been chosen. In this paper, we explicitly define the eight-tetrahedra longest-edge (8T-LE) partition of right-type tetrahedra and prove that for any regular right-type tetrahedron t, the iterative 8T-LE partition of t yields a sequence of tetrahedra similar to the former one. Furthermore, based on the Kuhn-type triangulations, the 8T-LE partition commutes with certain refinements based on the canonical boxel partition of a cube and its Kuhn triangulation.

Keywords: Eight-tetrahedra longest-edge partition; Right-type tetrahedron; Kuhn triangulation

1. Introduction

Triangles and tetrahedra have been widely used for local adaptive refinement, and several bisection-based algorithms in two [1–3] and three [4–7] dimensions have been presented in recent years. The eight-tetrahedra longest-edge (8T-LE) partition of a general tetrahedron has been introduced in [6]. We focus in this paper on the 8T-LE partition of a special type of tetrahedra, called regular right-type tetrahedra. These tetrahedra have four right isosceles triangles as faces. For any regular right-type initial tetrahedron t, we prove that the iterative 8T-LE partition of t yields a sequence of regular right-type tetrahedra similar to t.

Definition 1 (*Simplex*). A closed subset $T \subset \mathbb{R}^n$ is called a (k)-*simplex*, $0 \leq k \leq n$, if T is the convex linear hull of $k+1$ vertices $\boldsymbol{x}^{(0)}, \boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)} \in \mathbb{R}^n$, and it will be denoted by $T = [\boldsymbol{x}^{(0)}, \boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}]$.

If $k = n$ then T is simply called *simplex* or *triangle* in $\mathbb{R}^n$. In what follows, (2)-simplices and (3)-simplices are also called triangles and tetrahedra respectively.

Definition 2 (*Similarity Classes*). Two simplices $t, t' \in \mathbb{R}^n$ are called *similar* or *congruent* to each other if there exists a translation vector $\boldsymbol{a} \in \mathbb{R}^n$, a scaling factor $c > 0$, and an orthogonal matrix $\mathbf{Q} \in \mathbb{R}^{n \times n}$ such that

$$t' = \boldsymbol{a} + c\mathbf{Q}t. \quad (1)$$

In this case t and t' are elements of the same *similarity class*.

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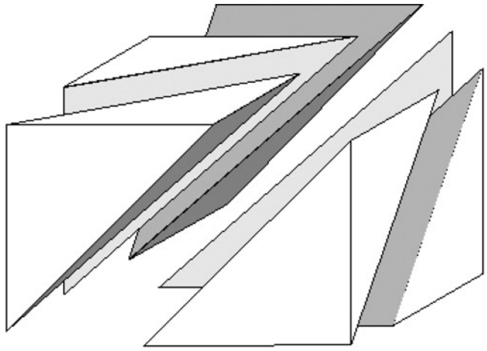

Fig. 1. Kuhn triangulation of the unit cube into six tetrahedra (exploded view).

Symbol " =" in Eq. (1) must be understood in the sense of sets, so the similarity class of a simplex is independent of its vertex ordering.

Definition 3 (*Conforming Triangulation*). Let Ω be any bounded domain in $\mathbb{R}^2$, or $\mathbb{R}^3$ with non-empty interior and polygonal boundary $\partial\Omega$, and consider a partition of Ω into a set of simplices (triangles or tetrahedra) $\tau = \{t_1, t_2, t_3, \ldots, t_n\}$, such that any adjacent simplex elements share an entire face or edge or a common vertex. Then we say that τ is a conforming simplex mesh or a conforming triangulation for Ω.

In general, from an initial tetrahedral mesh τ_0, by applying a bisection-based (local) refinement a sequence of finer triangulations $\{\tau_0, \tau_1, \ldots, \tau_n\}$ verifying the following conditions is obtained:

(1) Nestedness: Each element $t \in \tau_k$, $k > 0$ is covered by exactly one element $t' \in \tau_{k-1}$.
(2) Conformity: Each triangulation τ_k is conforming, which means that the intersection of any two tetrahedra in τ_k is either empty, a common face, a common edge or a common vertex.
(3) Non-degeneracy: The interior angles of all the elements are uniformly bounded away from zero, for any triangulations obtained by (local) refinement.

Note that in the case of (local) refinement based on edge-bisection, any corner of any element $t \in \tau_k$, $k > 0$ is either a corner or an edge mid-point of some element $t' \in \tau_{k-1}$.

We define the *Kuhn-type triangulation* of the unit cube $\mathcal{C} = [0, 1]^3$ with vertex $\mathbf{x}_0 \in \{0, 1\}^3$, according to [8]. Notice that since $\mathbf{x}_0 \in \{0, 1\}^3$, and the center of unit cube $\mathcal{C}$ is $B = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^{\mathrm{T}}$, then the opposite vertex of $\mathbf{x}_0$ is $\mathbf{x}_1 = \mathbf{x}_0 + 2\overrightarrow{\mathbf{x}_0 B}$. For example, if $\mathbf{x}_0 = (0, 0, 0)^{\mathrm{T}}$, then $\mathbf{x}_1 = (1, 1, 1)^{\mathrm{T}}$.

Definition 4 (*Kuhn-Type Triangulation*). Given cube $\mathcal{C} = [0, 1]^3$, and vertex $\boldsymbol{x}_0 \in \mathcal{C}$, with opposite vertex $\boldsymbol{x}_1$, the Kuhn triangulation of $\mathcal{C}$ is given by the six tetrahedra all sharing edges of vertices $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$, obtained as follows. For each permutation $\pi \in S_3$, tetrahedron $t_\pi = [\boldsymbol{x}_\pi^{(0)}, \boldsymbol{x}_\pi^{(1)}, \boldsymbol{x}_\pi^{(2)}, \boldsymbol{x}_\pi^{(3)}]$ is defined to be the closed convex hull of the vertices:

$$\boldsymbol{x}_\pi^{(0)} = \boldsymbol{x}^0, \qquad \boldsymbol{x}_\pi^{i} = \boldsymbol{x}_\pi^{i-1} + (-1)^{x^0_{\pi(i)}} \mathrm{e}^{\pi(i)}, \quad i = 1, 2, 3, \tag{2}$$

where e^j denotes the j-th standard unit vector in $\mathbb{R}^3$, x^0_j the coordinate j of the initial vertex $\boldsymbol{x}^0$. This triangulation will be denoted by $\mathcal{K}_{\boldsymbol{x}^0}(\mathcal{C}) := \{t_\pi / \pi \in S_3\}$.

The definition of the convex hull implies the representation

$$t_\pi = \{\boldsymbol{x} \in \mathcal{C} / 0 \le x_{\pi(3)} \le x_{\pi(2)} \le x_{\pi(1)} \le 1\} \quad \pi \in S_3. \tag{3}$$

Let $\boldsymbol{x}^0$ be a vertex of the unit cube $\mathcal{C}$, and $\mathcal{K}_{\boldsymbol{x}^0}(\mathcal{C})$ the associated Kuhn triangulation. One may easily verify that for $\pi \neq \pi'$ the intersection of t_π and $t_{\pi'}$ is a common lower dimensional sub-simplex, and thus the triangulation is conforming. Fig. 1 shows the triangulation with vertex $\boldsymbol{x}^0 = (0, 0, 0)^{\mathrm{T}}$ at the front left corner.

Note that the Kuhn triangulation of a cube is determined by the interior edge, of vertices $\boldsymbol{x}^0$ and $\boldsymbol{x}^1$ defined before, and the orthogonal projections of this edge over the faces of the cube. In fact the set of the edges of the unit cube,

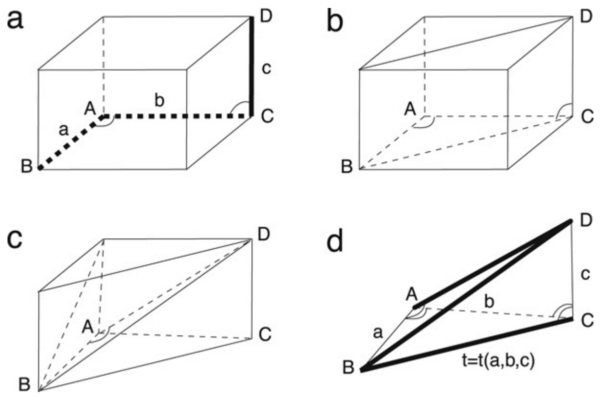

Fig. 2. Initial orthohedron $\mathcal{P}$ and construction of right-type tetrahedron $t = [ABCD] = t(a, b, c)$.

the interior edge $\boldsymbol{x}^0\boldsymbol{x}^1$, and the orthogonal projections on the faces constitute the set of one-dimensional simplices of the three-dimensional triangulation, and the one-dimensional simplicial complex determines the three-dimensional triangulation [9].

2. The 8T-LE partition of right-type tetrahedra

The 8T-LE partition of a general tetrahedron is obtained by successive mid-point bisection: first, the initial tetrahedron is divided by its longest edge, then the two tetrahedra obtained are bisected by the longest edges of the two faces of the initial tetrahedron not sharing the primary edge. Finally, the four tetrahedra are bisected by the mid-point of the common edge with the initial tetrahedron. A detailed explanation of the 8T-LE partition, and the associated local tetrahedral refinement can be found in [6].

A tetrahedron t is said to be a right-type tetrahedron if its four faces are right triangles. In such a tetrahedron t, there are three mutually perpendicular edges which do not pass through the same vertex, and are called legs of t. One of them has one vertex in common with each of the other legs. This leg is called the central leg and the others are the extreme legs. If the three legs are of the same length, the right-type tetrahedron will be called a regular right tetrahedron. The legs define, by parallelism, a unique orthohedron $\mathcal{P}$ that we call the orthohedron-hull of t, such that $t \subset \mathcal{P}$, the vertices of t are also vertices of $\mathcal{P}$, and the longest edge of t is an internal diagonal of $\mathcal{P}$. An example of a right-type tetrahedron t is shown in Fig. 2. The four faces of t are right-angled triangles, where the legs are highlighted in bold in Fig. 2(a). The edges out of the legs are called hypotenuses because they are the hypotenuses of the respective right triangular faces (in bold in Fig. 2(d)).

Note that the legs of a right-type tetrahedron are also the legs of the four faces of the tetrahedron. Moreover, the length and relative position of the legs determine the shape of any right-type tetrahedron. Let t be a right-type tetrahedron with legs a, b, and c, such that b is between a and c; then $t = t(a, b, c)$ will denote the tetrahedron t and at the same time the class of the tetrahedra similar to t.

Theorem 5. *Two right-type tetrahedra $t(a, b, c)$ and $t'(a', b', c')$ are similar to each other if and only if their extreme legs are in the same ratio as the central legs. That is, either $\frac{b}{b'} = \frac{a}{a'} = \frac{c}{c'}$, or $\frac{b}{b'} = \frac{a}{c'} = \frac{c}{a'}$.* □

Since the longest edges of the faces of any right-type tetrahedron are the hypotenuses, the 8T-LE partition of a right-type tetrahedron t can be described as an explicit function of its vertices. To this end, consider tetrahedron $t = [\boldsymbol{x}^{(0)}, \boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \boldsymbol{x}^{(3)}]$, such that $\boldsymbol{x}^{(0)}\boldsymbol{x}^{(1)}$, $\boldsymbol{x}^{(1)}\boldsymbol{x}^{(2)}$, and $\boldsymbol{x}^{(2)}\boldsymbol{x}^{(3)}$ are the legs of t. Observe that the primary edge of t is $\boldsymbol{x}^{(0)}\boldsymbol{x}^{(3)}$, and the secondary edges are $\boldsymbol{x}^{(1)}\boldsymbol{x}^{(3)}$ and $\boldsymbol{x}^{(0)}\boldsymbol{x}^{(2)}$. For $0 \le i, j \le 3, i \ne j$ we define $\boldsymbol{x}^{(ij)} := (\boldsymbol{x}^{(i)} + \boldsymbol{x}^{(j)})/2$, the edge mid-point of $\boldsymbol{x}^{(i)}$ and $\boldsymbol{x}^{(j)}$. The 8T-LE partition of a right tetrahedron t can be formulated as follows:
Algorithm 8T-LE partition of right tet (t)

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divide $t = [\boldsymbol{x}^{(0)}, \boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \boldsymbol{x}^{(3)}]$ *into the subtetrahedra* t_i, $1 \le i \le 8$, *given by*

$t_1 := [\boldsymbol{x}^{(0)}, \boldsymbol{x}^{(01)}, \boldsymbol{x}^{(02)}, \boldsymbol{x}^{(03)}]$, $\quad t_5 := [\boldsymbol{x}^{(2)}, \boldsymbol{x}^{(02)}, \boldsymbol{x}^{(12)}, \boldsymbol{x}^{(03)}]$,
$t_2 := [\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(01)}, \boldsymbol{x}^{(02)}, \boldsymbol{x}^{(03)}]$, $\quad t_6 := [\boldsymbol{x}^{(2)}, \boldsymbol{x}^{(12)}, \boldsymbol{x}^{(03)}, \boldsymbol{x}^{(13)}]$,

$$t_3 := [\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(02)}, \boldsymbol{x}^{(12)}, \boldsymbol{x}^{(03)}], \qquad t_7 := [\boldsymbol{x}^{(2)}, \boldsymbol{x}^{(03)}, \boldsymbol{x}^{(13)}, \boldsymbol{x}^{(23)}],$$
$$t_4 := [\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(12)}, \boldsymbol{x}^{(13)}, \boldsymbol{x}^{(03)}], \qquad t_8 := [\boldsymbol{x}^{(3)}, \boldsymbol{x}^{(03)}, \boldsymbol{x}^{(13)}, \boldsymbol{x}^{(23)}].$$

}

3. Main result

Theorem 6. *Let t be a right-type tetrahedron with equal length legs. Then, after applying the 8T-LE partition to t we obtain eight tetrahedra also of right type which constitute a conforming and non-degenerate triangulation. Moreover, in this case, all the elements are similar to the initial tetrahedron t.*

Proof. Non-degenerate triangulation follows from the last statement. We follow an argument similar to that of Bey in [10, Theorem 1, pages 373–375]. The proof is based on dissections of the unit cube $\mathcal{C} = [0, 1]^3$ in six tetrahedra passing into each other by permutation of their coordinates.

Let $\tau_0 = \mathcal{K}_{\boldsymbol{x}^0}(\mathcal{C})$ be the Kuhn triangulation of the unit cube $\mathcal{C} = [0, 1]^3$ with vertex $\boldsymbol{x}^0 = (0, 0, 0)^{\mathrm{T}}$. Another triangulation τ_1 of $\mathcal{C}$ can be defined in the following way: Let $\mathcal{B}$ be the canonical subdivision of $\mathcal{C}$ into eight sub-cubes of edge length $\frac{1}{2}$, that is

$$\mathcal{B} = \{\mathcal{C}_{\boldsymbol{x}} / \boldsymbol{x} \in \{0, 1\}^3\} \tag{4}$$

where, for each $\boldsymbol{x} \in \{0, 1\}^3$, $\mathcal{C}_{\boldsymbol{x}}$ is given by

$$\mathcal{C}_{\boldsymbol{x}} := \boldsymbol{x} + \frac{1}{2}\sigma_{\boldsymbol{x}}(\mathcal{C}) := \left\{\boldsymbol{x} + \frac{1}{2}\sigma_{\boldsymbol{x}}(\boldsymbol{x}') / \boldsymbol{x}' \in \mathcal{C}\right\} \tag{5}$$

where $\sigma_{\boldsymbol{x}}$ is the composition of mirror reflections by the coordinate planes given by the diagonal matrix Diag $((-1)^{x_1}, (-1)^{x_2}, (-1)^{x_3})$. The Kuhn triangulation of any sub-cube $\mathcal{C}_{\boldsymbol{x}}$ is given by selecting the pivot vertex of the triangulation. If we select as pivot vertex in each sub-cube $\mathcal{C}_{\boldsymbol{x}}$, vertex $\boldsymbol{x}$, or, equivalently, the vertex opposite to $\boldsymbol{x}$ in the sub-cube, which is precisely $B = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^{\mathrm{T}}$, we obtain a triangulation of the unit cube $\mathcal{C}$,

$$\tau_1 = \bigcup_{i=0}^{7} \mathcal{K}_{\boldsymbol{x}^i}(\mathcal{C}_{\boldsymbol{x}^i}).$$

We shall prove that triangulation τ_1 is precisely the 8T-LE global refinement of triangulation $\tau_0 = \mathcal{K}_{\boldsymbol{x}^0}(\mathcal{C})$. Since the projections of triangulation τ_1 on the faces of the sub-cubes are always conforming the triangulation obtained is conforming.

We now show that τ_1 is a refinement of τ_0 in the sense of nestedness. So, for any given element $t_1 \in \tau_1$ we should find an element $t_0 \in \tau_0$ such that $t_1 \subset t_0$.

Let element $t_1 \in \mathcal{K}_B(\mathcal{C}_{\boldsymbol{x}})$, for a fixed vertex $\boldsymbol{x} \in \mathcal{C}$. Then element $t_1 = \boldsymbol{x} + \sigma_{\boldsymbol{x}}(t_\pi)$ for the corresponding mirror reflection $\sigma_{\boldsymbol{x}}$ determined by vertex $\boldsymbol{x} \in \mathcal{C}$, and $\pi \in S_3$; we are looking for a permutation $\pi^\star = \pi^\star(\boldsymbol{x}, \pi)$ such that $t_1 = \boldsymbol{x} + \sigma_{\boldsymbol{x}}(t_\pi) \subset t_{\pi^\star} = t_0 \in \tau_0$.

Let $0 \le k \le 3$ be the number of entries x_i of $\boldsymbol{x}$ with $x_i = 1$. Then there are k unique indices $i_1, \ldots, i_k \in \{1, 2, 3\}$ satisfying

$$1 \le i_1 < \cdots < i_k \le 3, \quad x_{\pi(i_1)} = \cdots = x_{\pi(i_k)} = 1. \tag{6}$$

Here and in the following, for the case $k = 0$ and $k = 3$ we skip over those parts of the corresponding (in)equalities that make no sense. We now define $\pi^\star$, taking into account the number of mirror reflections which are necessary to take point $(0, 0, 0)$ to the vertex $\boldsymbol{x}$. Namely, there are the following instances:

(a) If $\boldsymbol{x} = (0, 0, 0)$, then $k = 0$ and $\pi^\star = \pi$.
(b) If $k = 1$, there is an index i_1 such that $x_{\pi(i_1)} = 1$. We define $\pi^\star(1) = \pi(i_1)$.
(c) If $k = 2$, there are two indices i_1, and i_2 such that $i_1 < i_2$ and $x_{\pi(i_1)} = x_{\pi(i_2)} = 1$. We define $\pi^\star(1) = \pi(i_2)$, and $\pi^\star(2) = \pi(i_1)$.
(d) Finally, if $k = 3$, $\boldsymbol{x} = (1, 1, 1)$, then we define $\pi^\star(1) = \pi(i_3)$, $\pi^\star(2) = \pi(i_2)$, and $\pi^\star(3) = \pi(i_1)$.

Table 1
Explicit definition of permutations $\pi^\star = \pi^\star(\boldsymbol{x}, \pi)$

	$\pi(1,2,3)$	$\pi(1,3,2)$	$\pi(2,1,3)$	$\pi(2,3,1)$	$\pi(3,1,2)$	$\pi(3,2,1)$
$\boldsymbol{x}_0(0,0,0)$	$\pi^\star(1,2,3)$	$\pi^\star(1,3,2)$	$\pi^\star(2,1,3)$	$\pi^\star(2,3,1)$	$\pi^\star(3,1,2)$	$\pi^\star(3,2,1)$
$\boldsymbol{x}_1(1,0,0)$	$\pi^\star(1,2,3)$	$\pi^\star(1,3,2)$	$\pi^\star(1,2,3)$	$\pi^\star(1,2,3)$	$\pi^\star(1,3,2)$	$\pi^\star(1,3,2)$
$\boldsymbol{x}_2(0,1,0)$	$\pi^\star(2,1,3)$	$\pi^\star(2,1,3)$	$\pi^\star(2,1,3)$	$\pi^\star(2,3,1)$	$\pi^\star(2,3,1)$	$\pi^\star(2,3,1)$
$\boldsymbol{x}_3(1,1,0)$	$\pi^\star(2,1,3)$	$\pi^\star(2,1,3)$	$\pi^\star(1,2,3)$	$\pi^\star(1,2,3)$	$\pi^\star(2,1,3)$	$\pi^\star(1,2,3)$
$\boldsymbol{x}_4(0,0,1)$	$\pi^\star(3,1,2)$	$\pi^\star(3,1,2)$	$\pi^\star(3,2,1)$	$\pi^\star(3,2,1)$	$\pi^\star(3,1,2)$	$\pi^\star(3,2,1)$
$\boldsymbol{x}_5(0,1,1)$	$\pi^\star(3,2,1)$	$\pi^\star(2,3,1)$	$\pi^\star(3,2,1)$	$\pi^\star(3,2,1)$	$\pi^\star(2,3,1)$	$\pi^\star(2,3,1)$
$\boldsymbol{x}_6(1,0,1)$	$\pi^\star(3,1,2)$	$\pi^\star(3,1,2)$	$\pi^\star(3,1,2)$	$\pi^\star(1,3,2)$	$\pi^\star(1,3,2)$	$\pi^\star(1,3,2)$
$\boldsymbol{x}_7(1,1,1)$	$\pi^\star(3,2,1)$	$\pi^\star(2,3,1)$	$\pi^\star(3,1,2)$	$\pi^\star(1,3,2)$	$\pi^\star(2,1,3)$	$\pi^\star(1,2,3)$

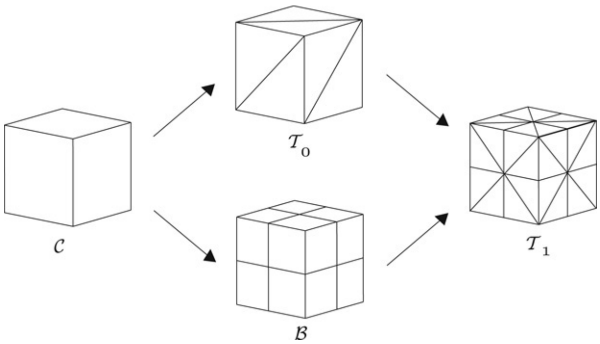

Fig. 3. Two ways of generating triangulation τ_1.

However the remaining $3-k$ indices $i_{k+1}, \ldots, i_3 \in \{1,2,3\}$ can be ordered such that

$$1 \le i_{k+1} < \cdots < i_3 \le 3, \quad x_{\pi(i_{k+1})} = \cdots = x_{\pi(i_3)} = 0. \tag{7}$$

For these remaining $3-k$ indices, if any, we define $\pi^\star(j) = \pi(i_j)$, $3-k \le j \le 3$. From the right hand sides of (6) and (7), we conclude that

$$x_{\pi^\star(1)} = \cdots = x_{\pi^\star(k)} = 1, \qquad x_{\pi^\star(k+1)} = \cdots = x_{\pi^\star(3)} = 0. \tag{8}$$

The previously established relation between permutation $\pi^\star$ and permutation π and pivot vertex $\boldsymbol{x}$ is explicitly shown in Table 1.

Further, consider any $\xi = (\xi_1, \xi_2, \xi_3)^{\mathrm{T}} \in t_\pi \in \mathcal{K}_{\boldsymbol{x}^0}(\frac{1}{2}\mathcal{C})$, the Kuhn triangulation of the half-unit cube $\frac{1}{2}\mathcal{C} = [0, \frac{1}{2}]^3$ with vertex $\boldsymbol{x}^0 = (0,0,0)^{\mathrm{T}}$. We have $0 \le \xi_{\pi(3)} \le \xi_{\pi(2)} \le \xi_{\pi(1)} \le 1/2$. Using the left hand sides of (6) and (7), we obtain

$$\frac{-1}{2} \le \xi_{\pi^\star(k)} \le \cdots \le \xi_{\pi^\star(1)} \le 0, \qquad 0 \le \xi_{\pi^\star(3)} \le \cdots \le \xi_{\pi^\star(k+1)} \le \frac{1}{2}. \tag{9}$$

Then, combining (8) and (9) with (3), for any vertex of the initial cube $\boldsymbol{x} \in \{0,1\}^3$ and the corresponding composition of mirror reflections $\sigma_{\boldsymbol{x}}(\xi)$, we get $\boldsymbol{x} + \sigma_{\boldsymbol{x}}(\xi) \in t_{\pi^\star}$ which proves $\boldsymbol{x} + \sigma(t_\pi) = t_{\boldsymbol{x},\pi} \subset t_{\pi^\star}$. Of course, by construction, any corner of $t_{\boldsymbol{x},\pi}$ corresponds to either a corner or an edge mid-point of $t_{\pi^\star}$ and thus τ_1 is in fact a refinement of τ_0 in the sense of condition (1).

At this point we have shown the existence of a refinement method for the elements of the Kuhn triangulation τ_0 of $\mathcal{C}$. This method yields the same triangulation τ_1 as is obtained if we first subdivide $\mathcal{C}$ into eight sub-cubes $\mathcal{B} \in \mathcal{C}$ and these again by a Kuhn-type triangulation with vertex the center of the initial cube. Fig. 3 shows a commutative diagram illustrating these equivalent ways of generating triangulation τ_1.

We now want to show that τ_1 is exactly the triangulation which is generated if the partition 8T-LE is applied to all the elements of $t_\pi \in \tau_0$, provided their vertices are numbered according to (1). Therefore we first consider the *reference element* $t_0 := t_{\pi_{id}} = t_{\{1,2,3\}}$ with corners $\boldsymbol{x}^{(0)} = (0,0,0)^{\mathrm{T}}$, $\boldsymbol{x}^{(1)} = (1,0,0)^{\mathrm{T}}$, $\boldsymbol{x}^{(2)} = (1,1,0)^{\mathrm{T}}$, and $\boldsymbol{x}^{(3)} = (1,1,1)^{\mathrm{T}}$.

Using the Algorithm *8T-LE partition* to refine t_0, it can be easily verified that the son elements $t_{0,j}$, for $1 \le j \le 8$, can be represented by

$$t_{0,j} = \boldsymbol{x}^i + \frac{1}{2}\sigma_{i,j}(t_0), \quad 1 \le i, j \le 8 \tag{10}$$

where $\sigma_{i,j}$ is the mirror reflection determined by the non-zero coordinates of vertex $\boldsymbol{x}^i$. These mirror reflections are given in the following:

If the subsequent order of son elements $t_{0,j}$ corresponds to the formulation of the 8T-LE partition algorithm in Section 2, the start vertices $\boldsymbol{x}^i$ and mirror reflections, or permutations $\sigma_{i,j}$, $1 \le i, j \le 8$ are given by

$$\begin{aligned} &\boldsymbol{x}^1 = (0, 0, 0)^{\mathrm{T}} \\ &\boldsymbol{x}^2 = \boldsymbol{x}^3 = \boldsymbol{x}^4 = (1, 0, 0)^{\mathrm{T}} \\ &\boldsymbol{x}^5 = \boldsymbol{x}^6 = \boldsymbol{x}^7 = (1, 1, 0)^{\mathrm{T}} \\ &\boldsymbol{x}^8 = (1, 1, 1)^{\mathrm{T}} \end{aligned} \tag{11}$$

and

$$\begin{aligned} &\sigma_{1,1} = \mathrm{id} = \mathrm{Diag}(1, 1, 1); \qquad \sigma_{2,1} = \mathrm{Diag}(-1, 1, 1); \\ &\sigma_{3,2} = \mathrm{Diag}(-1, 1, 1) \circ (e_2, e_1, e_3); \\ &\sigma_{4,3} = \mathrm{Diag}(-1, 1, 1) \circ (e_2, e_3, e_1); \\ &\sigma_{5,1} = \mathrm{Diag}(1, 1, 1) \circ (e_2, e_1, e_3); \\ &\sigma_{6,2} = \mathrm{Diag}(-1, 1, 1) \circ (e_2, e_3, e_1); \\ &\sigma_{7,3} = \mathrm{Diag}(-1, -1, 1); \qquad \sigma_{8,1} = \mathrm{Diag}(-1, -1, -1) \end{aligned} \tag{12}$$

respectively. Representation (10) implies $t_{0,j} \in \tau_1$, for $1 \le j \le 8$. For $i \ne j$, (11) and (12) respectively show that either $\boldsymbol{x}^i \ne \boldsymbol{x}^j$ or $\sigma_i \ne \sigma_j$ is true. Therefore, $t_{0,i}$, $t_{0,j}$ correspond to different elements $t_{\boldsymbol{x}^i,\sigma_i}$, $t_{\boldsymbol{x}^j,\sigma_j}$, which are known to have mutually disjoint interiors. Furthermore, we conclude from (10) that the volumes of all son elements sum up to the volume of t_0, and thus the convexity of t_0 implies that the generated refinement of t_0 coincides with the one induced by τ_1.

To obtain the same result for the other elements in τ_0, we associate with each $\pi \in S_3$ the corresponding permutation matrix given by $P_\pi = \left(\delta_{i,\pi(j)}\right)_{i,j=1}^3$. We then have $t_\pi = P_\pi(t_0)$ and in particular for the corners $\boldsymbol{x}_\pi^j = P_\pi(\boldsymbol{x}_{id}^j)$, $0 \le j \le 3$. Applying algorithm *8T-LE partition* to t_π yields the sons $t_{\pi,i} = P_\pi(t_{0,i})$, $0 \le i \le 8$. Denoting by $\pi \circ \pi_i$ the composition of π, π_i within S_3, and using the fact that the associated permutation matrix is given by $P_{\pi\circ\pi_i} = P_\pi \circ P_{\pi_i}$, the analogue to (10) is the next equation:

$$t_{\pi,i} = P_\pi(t_{0,i}) = P_\pi(\boldsymbol{x}^i) + \frac{1}{2}t_{\pi\circ\pi_i}. \tag{13}$$

Now $P_\pi(\boldsymbol{x}^i) \in \{0, 1\}^3$ implies $t_{\pi,i} \in \tau_1$ for each $1 \le i \le 8$, $\pi \in S_3$. Using the argumentation from above, it follows that the generated refinement of t_π coincides with the one induced by τ_1 by algorithm *8T-LE partition*.

In addition to (10), i.e. $t_{0,j} = \boldsymbol{x}^i + \frac{1}{2}\,\sigma_{i,j}(t_0)$, for $1 \le i, j \le 8$, we observe that the vertex numbering assigned to $t_{0,j}$ by the refinement of t_{π_i}, i.e. the i-th corner of $t_{0,i}$, is given by $\boldsymbol{x}^i + 1/2\boldsymbol{x}^j_{\pi_i}$. This property is preserved under permutation and remains valid for any element of τ_1. If now algorithm *8T-LE partition* is recursively applied to the elements of τ_1, it follows by induction that the generated triangulations τ_k, $k \ge 0$, of $\mathcal{C}$ are given by

$$\tau_k = \{t_{\boldsymbol{x},\pi} = \boldsymbol{x} + 2^{-k}t_\pi \mid \boldsymbol{x} \in \{0, 1\cdot 2^{-k}, \ldots, (2^k - 1)\cdot 2^{-k}\}^3, \pi \in S_3\}, \tag{14}$$

and, thus, it can also be obtained by first dividing $\mathcal{C}$ into 8^k sub-cubes of edge length 2^{-k}, which in turn are subdivided by the corresponding Kuhn-type triangulation. These arguments complete the proof. □

4. Conclusions

In this paper we have proved that for any regular right-type tetrahedron t, the iterative eight-tetrahedra longest-edge (8T-LE) partition of t yields a sequence of tetrahedra similar to the former one. Furthermore, based on the Kuhn-type triangulations, the 8T-LE partition commutes with certain refinements based on the canonical boxel partition of a cube and its Kuhn triangulations.

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