# Billiards in a circle with trajectories circumscribing a triangle 

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#### Abstract

We consider a bar billiards problem for a triangle in the unit circle. For the point on the unit circle, we construct a line from it in a counterclockwise direction tangent to the triangle, and examine a map corresponding to the point of intersection with the circle. For the rotation number $\rho$ of this map, we give $\frac{1}{3} \leq \rho<\frac{1}{2}$ and necessary and sufficient conditions for $\rho=\frac{1}{3}$, which is related to an ellipse. We give an application to elementary geometry.


## 1 Introduction

Let $S^{1}$ be the unit circle and $C$ be a circle lying inside $S^{1}$. For $v \in S^{1}$, there exist two points $u_{1}, u_{2} \in S^{1}$ such that the segment $v u_{i}$ is tangent to $C$ for $i=1,2$, and $u_{1}$ is closer to $v$ in a counterclockwise direction than $u_{2}$. Then, the map $f: v \rightarrow u_{1}$ is a homeomorphism on $S^{1}$. A trajectory related to $\left(f^{n}(v)\left(=v_{n}\right)\right.$ for $n=0,1,2, \ldots$ (see Figure 1 (a)) has fascinated many researchers, since the finding of Poncelet porism (cf. [4],[8]). Even though the rotation number is rational, Poncelet porism states that $f$ is conjugated to the rotation. In a generalized situation, Mozgawa [6] considered such a billiard problem and obtained a theorem similar to Poncelet porism by using two ovals instead of circles. Mozgawa et al. [2], [6] called such a billiard

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problem a bar billiard. Cima et al. [3] considered a bar billiard problem, where the unit circle is placed inside the the curve $\left\{x^{2 m}+y^{2 m}=2\right\}, m \in \mathbb{Z}_{>0}$, and showed the rotation number is $\frac{1}{4}$ and the map associated with the billiard problem is not conjugated to a rotation except for $m=1$. In this study, we analyze a bar billiard problem, in which a triangle, rather than an oval, is put inside the unit circle (see Figure $1(b)$ ). We provide certain conditions to ensure that the rotation number is $\frac{1}{3}$. When the rotation number is $\frac{1}{3}$, we also show the dynamics of the unit circle. The dynamics are not conjugate to a rotation. If the distance between one vertex and the line connecting the other two vertices of a triangle within the unit circle is larger than or equal to a certain length, then the rotation number attributed with the bar billiard is $\frac{1}{3}$, where the distance between two points is the distance related to the Beltrami-Klein model. For $P, Q$ in the unit circle, the set of $R$ such that the rotation number related to the bar billiard with $\triangle P Q R$ is $\frac{1}{3}$ is on or outside the circumference of an ellipse (see Figure 2). As an example of application to elementary geometry, we show that wherever an equilateral triangle with a circumradius of $\frac{1}{2}$ or larger than $\frac{1}{2}$ is put within the unit circle, there is a triangle inscribed in the unit circle and circumscribing the triangle.

The paper is organized as follows. In Section 2, in the Poincaré hyperbolic disk, we discuss the conditions that for a given triangle there exists a triangle inscribed in the unit circle and circumscribing this triangle in the hyperbolic sense. In Section 3, we translate the results in Section 2 to the Beltrami-Klein model. Section 4 examines the rotation numbers associated with the bar billiards and the dynamics of the unit circle. In Section 5, we show how this can be applied to elementary geometry.

## 2 Hyperbolic Disk

Let $D_{P}$ denote the Poincaré hyperbolic disk $\left\{(x, y) \in \mathbb{R}^{2} \mid x^{2}+y^{2}<1\right\}$ and the metric $d s^{2}=4 \frac{d x^{2}+d y^{2}}{\left(1-x^{2}-y^{2}\right)^{2}} . d(P, Q)$ is defined as the hyperbolic distance for $P, Q \in D_{P}$, i.e.,

$$
d(P, Q):=\operatorname{arccosh}\left(1+\frac{2|P-Q|^{2}}{\left(1-|P|^{2}\right)\left(1-|Q|^{2}\right)}\right)
$$

where $|\cdot|$ is the Euclidean norm. For example, see [1], [5].


Figure 1: Trajectories circumscribing a circle or a triangle.


Figure 2: Ellipse associated with a point of rotation number $\frac{1}{3}$.

For $P, Q \in D_{P}$, we define

$$
\Delta(P, Q):=\log \left(\frac{e^{d(P, Q)}+1}{e^{d(P, Q)}-1}\right)\left(=\log \left(\operatorname{coth}\left(\frac{d(P, Q)}{2}\right)\right)\right) .
$$

Theorem 2.1. Let $\triangle P Q R$ be a triangle in $D_{P}$. Let $m \in \mathbb{Z}_{\geq 0}$ be the number of triangles inscribed in $S^{1}$ and circumscribing $\triangle P Q R$. Let $h$ be the length of the vertical line from $R$ to $P Q$. Then,

$$
m= \begin{cases}0 & \text { if } h<\Delta(P, Q) \\ 1 & \text { if } h=\Delta(P, Q) \\ 2 & \text { if } h>\Delta(P, Q)\end{cases}
$$

Proof. Now we consider $D_{P}$ and $S^{1}$ to be the sets in $\mathbb{C}$; i.e., $D_{P}=\{z \in$ $\mathbb{C}||z|<1\}$ and $S^{1}=\{z \in \mathbb{C}| | z \mid=1\}$. Let $H$ be the upper half plane $\{z \in \mathbb{C} \mid$ the imaginary part of $z$ is positive $\}$. Let $f: D_{P} \rightarrow H$ be the map $f(z):=\frac{i z+i}{1-z}$ for $z \in D_{P}$. Then, it is known that $f$ is an isometry from $D_{P}$ to $H$ with the metric $d s^{2}=\frac{d x^{2}+d y^{2}}{y^{2}}$. For $z_{1}, z_{2} \in H$, we denote $d^{\prime \prime}\left(z_{1}, z_{2}\right)$ by the distance from $z_{1}$ to $z_{2}$ related to the metric $d s$. We see that $S L(2, \mathbb{R})$ is an isometry group of $H$ and it acts transitively on the set $\left\{\left(z_{1}, z_{2}\right) \in H^{2} \mid d^{\prime \prime}\left(z_{1}, z_{2}\right)=c\right\}$, where $c$ can be any positive constant. As a result, $g \in S L(2, \mathbb{R})$ exists such that $g(f(P))=i$ and $g(f(Q))$ is on the imaginary axis. Let $k i=g(f(Q))$ and we assume $k>1$ without loss of generality. Since $P, Q$, and $R$ are not on any line in $D_{P}, g(f(R))$ is not on the imaginary axis. We may assume that the real part of $g(f(R))$ is positive without loss of generality. Let $u+v i=g(f(R))$ with $u, v>0$. In hyperbolic geometry, a line in $H$ is a semicircle that intersects the real axis perpendicularly. We consider the circle $C_{1}$ with the center $s$ and $-t, k i \in C_{1}$ for $s, t \in \mathbb{R}$ with $t>0$. Then, we have

$$
s^{2}+k^{2}=(s+t)^{2},
$$

which implies $s=\frac{k^{2}-t^{2}}{2 t}$. Therefore, $2 s+t=\frac{k^{2}}{t}$ is on $C_{1}$. Similarly, the circle $C_{2}$ whose center is on the real axis and through $-t$ and $i$ has the point $\frac{1}{t}$. Let $C_{3}$ be the circle whose center is on the real axis and which passes through $\frac{1}{t}$ and $\frac{k^{2}}{t}$. The following is the condition $u+v i \in C_{3}$ :

$$
\left(u-\frac{1+k^{2}}{2 t}\right)^{2}+v^{2}=\left(\frac{k^{2}-1}{2 t}\right)^{2}
$$

which implies

$$
\begin{equation*}
\left(u^{2}+v^{2}\right) t^{2}-\left(k^{2}+1\right) u t+k^{2}=0 . \tag{2.1}
\end{equation*}
$$

The discriminant of formula (2.1) for t is

$$
D=\left(k^{2}+1\right)^{2} u^{2}-4 k^{2}\left(u^{2}+v^{2}\right) .
$$

We recall the definition of $m$ as the number of $C_{3}$, which includes $u+v i$. Since $D=\left(\left(k^{2}-1\right) u-2 k v\right)\left(\left(k^{2}-1\right) u+2 k v\right)$ and $u, v>0$, we have

$$
m= \begin{cases}0 & \text { if } v>\left(\frac{k^{2}-1}{2 k}\right) u  \tag{2.2}\\ 1 & \text { if } v=\left(\frac{k^{2}-1}{2 k}\right) u \\ 2 & \text { if } v<\left(\frac{k^{2}-1}{2 k}\right) u\end{cases}
$$

We denote the set $\left\{x+y i \in H \left\lvert\, y=\left(\frac{k^{2}-1}{2 k}\right) x\right.\right\}$ by $L$. Let $\theta \in(0, \pi / 2)$ be the angle between the imaginary axis and $L$. We will calculate the length $d$ of a vertical line from each point of $L$ to the imaginary axis in the hyperbolic sense. Let $x+i y \in L$ with $r=\sqrt{x^{2}+y^{2}}$. Then, the length of the curve $r \sin w+i r \cos w(0 \leq w \leq \theta)$ is

$$
\begin{equation*}
d=\int_{0}^{\theta} \frac{d w}{\cos w}=\log \left(\frac{1+\tan (\theta / 2)}{1-\tan (\theta / 2)}\right) \tag{2.3}
\end{equation*}
$$

We remark that $d$ depends only on $\theta$. On the other hand, from the fact that $\tan \theta=\frac{2 k}{k^{2}-1}$ we see that $\theta=\pi-2 \arctan k$. Therefore, from (2.3), we have

$$
\begin{equation*}
d=\log \left(\frac{k+1}{k-1}\right) . \tag{2.4}
\end{equation*}
$$

The distance between $i$ and $k i$ is

$$
\begin{equation*}
\int_{1}^{k} \frac{d y}{y}=\log k \tag{2.5}
\end{equation*}
$$

From (2.4) and (2.5), we have

$$
\begin{equation*}
d=\Delta(P, Q) \tag{2.6}
\end{equation*}
$$

Similarly, we can consider the case of $u<0$. The theorem is derived from (2.2) and (2.6).

The $h$ in Theorem 2.1 is calculated by hyperbolic trigonometry as follows.
Proposition 2.2. Let $\triangle P Q R$ be a triangle in $D_{P}$. Let $h$ be the length of the vertical line from $R$ down to the line $P Q$. Then,

$$
h=\operatorname{arcsinh}\left(\frac{\sqrt{(-\cosh (\alpha-\gamma)+\cosh b)(\cosh (\alpha+\gamma)+\cosh \beta)}}{\sinh \gamma}\right)
$$

where $\alpha=d(Q, R), \beta=d(R, P)$ and $\gamma=d(P, Q)$.
Proof. Through the hyperbolic law of cosines [1], we see

$$
\begin{equation*}
\cos Q=\frac{\cosh \alpha \cosh \gamma-\cosh \beta}{\sinh \alpha \sinh \gamma} \tag{2.7}
\end{equation*}
$$

Through the law of right-angled triangles [1] , we see

$$
\begin{equation*}
\sinh h=\sinh \alpha \sin Q \tag{2.8}
\end{equation*}
$$

From (2.7) and (2.8), we have the proposition.
In the proof of Theorem [2.1, we obtain the set containing the points $R$ with $m=1$. As a result, we may express Theorem 2.1 geometrically as follows.

Theorem 2.3. Let $P, Q, R \in D_{P}$ be different from each other. Let $T_{1}, T_{2} \in$ $S^{1}$ be two points at infinity which intersect the hyperbolic line $P Q$. Let $C_{1}, C_{2}$ be two arcs in $D_{P}$ intersecting $S^{1}$ at $T_{1}, T_{2}$ with angle $2 \arctan \left(e^{d(P, Q)}\right)-\pi / 2$. Let $\Theta$ be the inner region bounded by $C_{1}, C_{2}, T_{1}$ and $T_{2}$. The definition of $m \in \mathbb{Z}$ is the same as that of Theorem 2.1. Then,

$$
m= \begin{cases}0 & \text { if } R \in \Theta \\ 1 & \text { if } R \in C_{1} \cup C_{2} \\ 2 & \text { if } R \in(\operatorname{cl} \Theta)^{c}\end{cases}
$$

See Figure 3.

Proof. We recall the proof of Theorem 2.1. In the case of $u<0, L^{\prime}$ is defined as $\left\{x+y i \in H \left\lvert\, y=-\frac{k^{2}-1}{2 k} x\right.\right\}$. We define $\Theta^{\prime}$ as the inner region bounded by $L, L^{\prime}$ and 0, i.e.,

$$
\Theta^{\prime}:=\left\{\left.x+y i \in H| | y\left|>\left(\frac{k^{2}-1}{2 k}\right)\right| x \right\rvert\,\right\} .
$$



Figure 3: Two circular $\operatorname{arcs} C_{1}$ and $C_{2}$.

Then, we have

$$
m= \begin{cases}0 & \text { if } g(f(R)) \in \Theta^{\prime}  \tag{2.9}\\ 1 & \text { if } g(f(R)) \in L \cup L^{\prime} \\ 2 & \text { if } g(f(R)) \in\left(\operatorname{cl}^{\prime}\right)^{c}\end{cases}
$$

At 0 and $\infty$, the angle between $L$ and the imaginary axis is $2 \arctan \left(e^{d(P, Q)}\right)-$ $\pi / 2$. The same is true for $L^{\prime}$. The theorem is derived from (2.9) and the preceding facts.

The arcs will be discussed in greater depth in Proposition 2.4, as they are required in the following chapter.

Proposition 2.4. Let $P, Q \in D_{P}$ be different from each other. Let $T_{1}, T_{2} \in$ $S^{1}(=\mathbb{R} / \mathbb{Z})$ be two points at infinity, which intersect the hyperbolic line $P Q$ with $T_{1}-T_{2} \in(0,1 / 2] \bmod 1$. Let $T_{3} \in S^{1}(=\mathbb{R} / \mathbb{Z})$ be the point such that $T_{1}+T_{2}=2 T_{3} \bmod 1$ and $T_{3}-T_{2} \in(0,1 / 2] \bmod 1$. Let $C_{1}, C_{2}$ be two arcs in $D_{P}$ intersecting $S^{1}$ at $T_{1}, T_{2}$ with angle $2 \arctan \left(e^{d(P, Q)}\right)-\pi / 2$ such that $C_{1}$ is closer to $T_{3}$ than $C_{2}$. Then, the center of $C_{1}$ is

$$
\frac{e^{2 d(P, Q)}-1}{u\left(e^{2 d(P, Q)}-1\right)-2 e^{d(P, Q)} \sqrt{1-u^{2}}} T_{3},
$$

and the center of $C_{2}$ is

$$
\frac{e^{2 d(P, Q)}-1}{u\left(e^{2 d(P, Q)}-1\right)+2 e^{d(P, Q)} \sqrt{1-u^{2}}} T_{3},
$$



Figure 4: Points $A, B, C$.
where $u$ is the length between the origin and the straight line $T_{1} T_{2}$ in the sense of the Euclidean norm.
Proof. We put $k=e^{d(P, Q)}$. Let $A=a T_{3}$ be the center of $C_{1}$ where $a \in \mathbb{R}$. See Figure 4. Let $\theta$ be $\arccos u$. We assume that $a>0$. Let $B$ be the intersection of line $O T_{3}$ and the line vertical to line $A T_{1}$ that passes through $T_{1}$. Let $C$ be the intersection of line $O T_{3}$ and the line vertical to the line $O T_{1}$ that passes through $T_{1}$. Since $\angle B T_{1} C=2 \arctan (k)-\pi / 2$ and $\angle T_{1} O A=\theta$, we have $\angle O A T_{1}=2 \arctan k-\pi / 2-\theta$. Therefore, $A T_{1}=\sqrt{1-u^{2}} / \sin (2 \arctan k-$ $\pi / 2-\theta)$, which implies

$$
\begin{aligned}
a=u+ & \frac{\sqrt{1-u^{2}}}{\tan (2 \arctan k-\pi / 2-\theta)} \\
& =\frac{k^{2}-1}{u\left(k^{2}-1\right)-2 k \sqrt{1-u^{2}}} .
\end{aligned}
$$

For the case of $a<0$ we have the same formula. We can prove the formula for $C_{2}$ in the same manner.

## 3 Beltrami-Klein Disk

In this section, we interpret the theorems as geometric properties in the Euclidean plane through the Beltrami-Klein disk model(for example, see [1],5]). Let $\Delta$ be the disk $\left\{(x, y) \in \mathbb{R}^{2} \mid x^{2}+y^{2}<1\right\}$. We define $G: D_{P} \rightarrow D$ by for $(x, y) \in D_{P}$

$$
G(x, y):=\left(\frac{2 x}{1+x^{2}+y^{2}}, \frac{2 y}{1+x^{2}+y^{2}}\right) .
$$



Figure 5: $d^{\prime}(P, Q)$.

Then, the inverse map of $G$ is given by for $(x, y) \in D$

$$
\begin{equation*}
G^{-1}(x, y)=\left(\frac{x}{1+\sqrt{1-x^{2}-y^{2}}}, \frac{y}{1+\sqrt{1-x^{2}-y^{2}}}\right) \tag{3.1}
\end{equation*}
$$

$G$ is naturally extended to the boundary of $D_{P} . D$ is a hyperbolic geometric model that is isomorphic to $D_{P}$ via the map $G$ and it is called the Bel-trami-Klein model. A line of $D$ in the hyperbolic sense is known to be a straight line segment with endpoints at the boundary of $D$. For $P, Q \in D$, we define $d^{\prime}(P, Q)$ as

$$
d^{\prime}(P, Q):=d\left(G^{-1}(P), G^{-1}(Q)\right)
$$

We remark that $d^{\prime}(P, Q)$ is defined more simply using points at infinity. Let $v_{1}, v_{2} \in S^{1}$ be two points at infinity which intersect the hyperbolic line $P Q$ (see Figure 5). Then, $d^{\prime}(P, Q)$ is given by

$$
\frac{1}{2}\left|\log \left(\frac{\left|v_{1} Q \| v_{2} P\right|}{\left|v_{1} P \| v_{2} Q\right|}\right)\right| .
$$

For $P, Q \in D$ we define $\Delta^{\prime}(P, Q)$ as

$$
\Delta^{\prime}(P, Q):=\Delta\left(G^{-1}(P), G^{-1}(Q)\right)
$$

We define $\delta(P, Q, R)$ as the minimum of $d^{\prime}(R, S)$, where $S$ are points on the line $P Q$. From Proposition $2.2 \delta(P, Q, R)$ is given by

$$
\operatorname{arcsinh}\left(\frac{\sqrt{(-\cosh (\alpha-\gamma)+\cosh b)(\cosh (\alpha+\gamma)+\cosh \beta)}}{\sinh \gamma}\right)
$$

where $\alpha=d^{\prime}(Q, R), \beta=d^{\prime}(R, P)$ and $\gamma=d^{\prime}(P, Q)$.
From Theorem 2.1 and Proposition [2.2, we have
Theorem 3.1. Let $\triangle P Q R$ be a triangle in $D$. Let $m \in \mathbb{Z}_{\geq 0}$ be the number of triangles inscribed in $S^{1}$ and circumscribing $\triangle P Q R$. Then,

$$
m= \begin{cases}0 & \text { if } \delta(P, Q, R)<\Delta^{\prime}(P, Q), \\ 1 & \text { if } \delta(P, Q, R)=\Delta^{\prime}(P, Q), \\ 2 & \text { if } \delta(P, Q, R)>\Delta^{\prime}(P, Q) .\end{cases}
$$

We give an example.
Example 3.1. Let $P=\left(-\frac{1}{4}, \frac{\sqrt{3}}{4}\right), Q=\left(-\frac{1}{4},-\frac{\sqrt{3}}{4}\right), R=\left(\frac{1}{2}, 0\right)$, and $S=$ $\left(-\frac{1}{4}, 0\right)$. We set $v_{1}=\left(-\frac{1}{4}, \frac{\sqrt{15}}{4}\right), v_{2}=\left(-\frac{1}{4},-\frac{\sqrt{15}}{4}\right), v_{3}=(1,0)$, and $v_{4}=$ $(-1,0)$, which are points at infinity. Then, $\triangle P Q R$ is an equilateral triangle in $D$. See Figure 6. Then,

$$
\begin{aligned}
d^{\prime}(P, Q) & =\frac{1}{2} \log \frac{\left|v_{1} Q\right|\left|P v_{2}\right|}{\left|v_{2} Q\right|\left|P v_{1}\right|}=\log \frac{\sqrt{5}+1}{\sqrt{5}-1} \\
\Delta^{\prime}(P, Q) & =\log \frac{e^{d^{\prime}(P, Q)}+1}{e^{d^{\prime}(P, Q)}-1}=\log \sqrt{5}
\end{aligned}
$$

Conversely, we have

$$
\delta(P, Q, R)=d^{\prime}(R, S)=\frac{1}{2} \log \frac{\left|v_{3} S\right|\left|R v_{4}\right|}{\left|v_{4} S\right|\left|R v_{3}\right|}=\log \sqrt{5} .
$$

Therefore, $m$ (defined in Theorem 3.1) is 1 .
Theorem 3.1 shows that the relationship between the values of $\delta(P, Q, R)$ and $\Delta^{\prime}(P, Q)$ is the same as that between $\delta(R, P, Q)$ and $\Delta^{\prime}(R, P)$, and that between $\delta(Q, R, P)$ and $\Delta^{\prime}(Q, R)$. We define $\omega(R, P, Q)$ as

$$
\omega(R, P, Q):=\frac{\delta(P, Q, R)+\delta(R, P, Q)+\delta(Q, R, P)}{\Delta^{\prime}(P, Q)+\Delta^{\prime}(R, P)+\Delta^{\prime}(Q, R)} .
$$

Immediately from Theorem 3.1, we have
Proposition 3.2. Let $\triangle P Q R$ be a triangle in $D$. Let $m \in \mathbb{Z}_{\geq 0}$ be the number of triangles, which are inscribed in $S^{1}$ and circumscribe $\triangle P Q R$. Then,

$$
m= \begin{cases}0 & \text { if } \omega(P, Q, R)<1 \\ 1 & \text { if } \omega(P, Q, R)=1 \\ 2 & \text { if } \omega(P, Q, R)>1\end{cases}
$$



Figure 6: $\triangle P Q R$ in Example3.1.

From Theorem 2.3 and Proposition [2.4, we have
Theorem 3.3. Let $\triangle P Q R$ be a triangle in $D$. Let $T_{1}, T_{2} \in S^{1}(=\mathbb{R} / \mathbb{Z})$ be two points at infinity, which intersect the hyperbolic line $P Q$ with $T_{1}-T_{2} \in$ $(0,1 / 2] \bmod 1$. Let $T_{3}, T_{4} \in S^{1}(=\mathbb{R} / \mathbb{Z})$ be the points such that $T_{1}+T_{2}=$ $2 T_{3} \bmod 1, T_{3}-T_{2} \in(0,1 / 2] \bmod 1$, and $T_{4}=T_{3}+\frac{1}{4} \bmod 1 . \operatorname{Let}\left(x_{1}, x_{2}\right)^{\prime}$ be the coordinates using the base $\left\{T_{3}, T_{4}\right\}$. Let $u$ be the $x_{1}$ coordinate of $P$. We define $a \in \mathbb{R}$ as

$$
\frac{e^{2 d^{\prime}(P, Q)}-1}{u\left(e^{2 d^{\prime}(P, Q)}-1\right)-2 e^{d^{\prime}(P, Q)} \sqrt{1-u^{2}}} .
$$

Let $E$ be the ellipse defined by

$$
\left\{\left(x_{1}, x_{2}\right)^{\prime} \left\lvert\, \frac{\left(x_{1}-C\right)^{2}}{A^{2}}+\frac{x_{2}^{2}}{B^{2}}=1\right.\right\}
$$

where

$$
\begin{aligned}
& A:=\frac{|a u-1| \sqrt{a^{2}-2 u a+1}}{\left(u^{2}+1\right) a^{2}-2 a u+1}\left(=\frac{2 k\left(k^{2}+1\right)\left(1-u^{2}\right)}{\left(k^{2}-2 k u+1\right)\left(k^{2}+2 k u+1\right)}\right), \\
& B:=\frac{\sqrt{a^{2}-2 u a+1}}{\sqrt{\left(u^{2}+1\right) a^{2}-2 a u+1}}\left(=\frac{\left(k^{2}+1\right) \sqrt{1-u^{2}}}{\sqrt{\left(k^{2}-2 k u+1\right)\left(k^{2}+2 k u+1\right)}}\right), \\
& C:=\frac{a^{2} u}{\left(u^{2}+1\right) a^{2}-2 a u+1}\left(=\frac{\left(k^{2}-1\right)^{2} u}{\left(k^{2}-2 k u+1\right)\left(k^{2}+2 k u+1\right)}\right),
\end{aligned}
$$



Figure 7: Ellipse E.
where $k:=e^{d^{\prime}(P, Q)}$. See Figure 7. Then, $E$ is included in $D \cup S^{1}$ and is tangent to $S^{1}$ at $T_{1}$ and $T_{2}$. Let $\Phi$ be the inner region bounded by $E$. The definition of $m \in \mathbb{Z}$ is the same as that of Theorem 3.1. Then,

$$
m= \begin{cases}0 & \text { if } R \in \Phi \\ 1 & \text { if } R \in E \\ 2 & \text { if } R \in(\operatorname{cl} \Phi)^{c} .\end{cases}
$$

Proof. For simplicity, we assume that $P=(u, p), Q=(u, q)$, where $0 \leq u<$ $1, p>q$. First, we show that a circle with a center on the $x$-axis maps to an ellipse with a center on the $x$-axis by the map $G$. We define circle $C(r, b)$ by $\left\{(x, y) \mid(x-b)^{2}+y^{2}=r^{2}\right\}$. The map $G$ is extended to the map $\bar{G}$ from $\mathbb{R}^{2}$ to $D \cup S^{1}$ using the same formula (3.1). Then, using simple calculations, $\bar{G}$ bijectively maps $C(r, b)$ to the ellipse $E(r, b)$ defined by

$$
\frac{\left(x-C^{\prime}\right)^{2}}{A^{\prime 2}}+\frac{y^{2}}{B^{\prime 2}}=1
$$

where

$$
\begin{aligned}
A^{\prime} & :=\frac{2 r\left(1+r^{2}-b^{2}\right)}{\left((r-b)^{2}+1\right)\left((r+b)^{2}+1\right)} \\
B^{\prime} & :=\frac{2 r}{\sqrt{(r-b)^{2}+1} \sqrt{(r+b)^{2}+1}}, \\
C^{\prime} & :=\frac{2 b\left(1-r^{2}+b^{2}\right)}{\left((r-b)^{2}+1\right)\left((r+b)^{2}+1\right)}
\end{aligned}
$$

For $V_{1}, V_{2} \in C(r, b)$, we denote $\operatorname{arc}\left(V_{1} V_{2}\right)$ by the arc of $C(r, b)$ that goes counterclockwise from $V_{1}$ to $V_{2}$. For $V_{1}, V_{2} \in E(r, b)$, we denote $\operatorname{arc}^{\prime}\left(V_{1} V_{2}\right)$ by the arc of $E(r, b)$ that goes counterclockwise from $V_{1}$ to $V_{2}$. We suppose that $C(r, b)$ intersects $S^{1}$ at $T_{1}=\left(u, \sqrt{1-u^{2}}\right), T_{2}=\left(u,-\sqrt{1-u^{2}}\right)$. Then, we have $r^{2}=1+b^{2}-2 b u, C(r, 0)=S^{1}$, and $C\left(r, \frac{1}{u}\right)$ is a line in the hyperbolic sense. We suppose $n \neq 0, \frac{1}{u}$. Then, we have

1) $\operatorname{arc}\left(T_{2} T_{1}\right)$ is mapped to $\operatorname{arc}^{\prime}\left(T_{2} T_{1}\right)$ by $\bar{G}$ if $b>\frac{1}{u}$ or $b<0$, and $\operatorname{arc}\left(T_{1} T_{2}\right)$ is mapped to $\operatorname{arc}^{\prime}\left(T_{1} T_{2}\right)$ by $\bar{G}$ if $0<b<\frac{1}{u}$, where for the case of $u=0$ we set $\frac{1}{u}:=\infty$.
2) Furthermore, we suppose that $C\left(r^{\prime}, b^{\prime}\right)$ intersects $S^{1}$ at $T_{1}, T_{2}$, and $b^{\prime}=$ $\frac{b}{2 b u-1}$. Then, $E(r, b)=E\left(r^{\prime}, b^{\prime}\right)$.

The proofs of 1) and 2) are left to the reader. Theorem [2.3, Proposition 2.4, 1), and 2) can be used to prove the theorem. In Proposition [2.4, we set $G^{-1}(P)$ (resp., $\left.G^{-1}(Q)\right)$ as $P$ (resp., $Q$ ). $C_{1}$ in Proposition 2.4, is the arc of $C\left(\sqrt{(a-u)^{2}+1-u^{2}}, a\right)$, and $C_{2}$ is the arc of $C\left(\sqrt{\left(a^{\prime}-u\right)^{2}+1-u^{2}}, a^{\prime}\right)$, where $a^{\prime}:=a /(2 a u-1)$. As a result of 1) and 2), we have completed the proof.

In Theorem 3.3 let $P=(u, p)^{\prime}$ and $Q=(u, q)^{\prime}$. we have

$$
d^{\prime}(P, Q)=\frac{1}{2}\left|\log \left(\frac{\left(p+\sqrt{1-u^{2}}\right)\left(-q+\sqrt{1-u^{2}}\right)}{\left(q+\sqrt{1-u^{2}}\right)\left(-p+\sqrt{1-u^{2}}\right)}\right)\right| .
$$

Then, the ellipse $E$ in Theorem 3.3 is given by

## Proposition 3.4.

$$
\left\{\left(x_{1}, x_{2}\right)^{\prime} \left\lvert\, \frac{\left(x_{1}-C\right)^{2}}{A^{2}}+\frac{x_{2}^{2}}{B^{2}}=1\right.\right\},
$$



Figure 8: Ellipse as envelope.
where

$$
\begin{aligned}
& A:=\sqrt{\frac{\left(p^{2}+u^{2}-1\right)\left(q^{2}+u^{2}-1\right)\left(p q+u^{2}-1\right)^{2}}{\left(p^{2}\left(q^{2}+u^{2}\right)-2 p q+\left(q^{2}-2\right) u^{2}+u^{4}+1\right)^{2}}}, \\
& B:=\sqrt{\frac{\left(p q+u^{2}-1\right)^{2}}{p^{2}\left(q^{2}+u^{2}\right)-2 p q+\left(q^{2}-2\right) u^{2}+u^{4}+1}}, \\
& C:=\frac{u(p-q)^{2}}{p^{2}\left(q^{2}+u^{2}\right)-2 p q+\left(q^{2}-2\right) u^{2}+u^{4}+1} .
\end{aligned}
$$

Remark 3.1. Proposition 3.4 can also be demonstrated without the use of hyperbolic geometry. Given two points $P=(u, p), Q=(u, q)$ in $D$, let us characterize the region $\mathcal{R}_{P, Q} \subset D$ such that for all $R \in \mathcal{R}_{P, Q}$ there exists a triangle $\triangle u_{1} u_{2} u_{3}$ inscribed in $S^{1}$ and circumscribing $\triangle P Q R$. If $\triangle u_{1} u_{2} u_{3}$ exists, we can infer that $u_{1}=(\cos \theta, \sin \theta)$ and $\overline{u_{1} u_{2}}, \overline{u_{1} u_{3}}$ include $P, Q$. Then, $R$ lies on the open segment $\sigma_{\theta}:=\overline{u_{2} u_{3}}$, and, for any point $R^{\prime} \in \sigma_{\theta}, \triangle u_{1} u_{2} u_{3}$ circumscribes $\triangle P Q R^{\prime}$. Thus, $\sigma_{\theta} \subset \mathcal{R}_{P, Q}$ and therefore, $\mathcal{R}_{P, Q}=\bigcup_{\theta \in[0,2 \pi]} \sigma_{\theta}$. As $\mathcal{R}_{P, Q}$ is a union of segments, its boundary component in $D$ is the envelope of the family of those segments (see Figure 8).

Let us now directly deduce the envelope equation. The segment $\sigma_{\theta}$ is
contained within the line described by the following equation:

$$
\begin{aligned}
F(x, y, \theta):= & \left(\left(u^{2}+1-p q\right) \cos \theta+(p+q) u \sin \theta-2 u\right) x \\
& +\left((p+q) u \cos \theta+\left(1+p q-u^{2}\right) \sin \theta-p-q\right) y \\
& -2 u \cos \theta-(p+q) \sin \theta+p q+u^{2}+1 \\
= & 0
\end{aligned}
$$

By eliminating the variable $\theta$ from $F(x, y, \theta)=0$ and $\frac{\partial}{\partial \theta} F(x, y, \theta)=0$, we obtain the same equation of ellipse as in Proposition 3.4.

## 4 Rotation Number

Let $\triangle P Q R$ be a triangle in $D$. For $v \in S^{1}$, there exist two points $v_{1}, v_{2} \in S^{1}$ such that the segment $v v_{i}$ is tangent to $\triangle P Q R$ for $i=1,2$. We may assume that $v_{1}$ is closer to $v$ in a counterclockwise direction than $v_{2}$. We define the transformation $\psi_{\triangle P Q R}$ on $S^{1}$ as the mapping $v$ to $v_{1}$. Let $\pi: \mathbb{R} \rightarrow S^{1}$ be the projection

$$
\pi(x):=x-\lfloor x\rfloor, \text { for } x \in \mathbb{R}
$$

where $\lfloor x\rfloor$ is an integral part of $x$. Let $\overline{\psi_{\triangle P Q R}}$ be the lift with $\overline{\psi_{\triangle P Q R}}(0) \in$ $(0,1)$. Since $\psi_{\triangle P Q R}$ is a homeomorphism of $S^{1}$, we consider the rotation number of $\psi_{\triangle P Q R}$, which is denoted by $\rho\left(\psi_{\triangle P Q R}\right)$, where for example see [7] for rotation numbers.

In this section we show that $\triangle P Q R$ is rather large, the rotation number is $1 / 3$, and we consider the trajectory $\left(\psi_{\triangle P Q R}\right)^{n}(v)$ for $n=0,1,2, \ldots$.

We require a lemma.
Lemma 4.1. Let $\triangle P Q R$ be a triangle in $D$. Let $v_{1}, v_{2} \in S^{1}$ be points at infinity, which are intersection points with the line $P Q$ and $S^{1}$. We suppose $v_{1}$ is closer to $P$ than $Q$ and $\psi_{\triangle P Q R}\left(v_{1}\right)=v_{2}$. Then, if $v_{1}^{\prime} \in \pi^{-1}\left(v_{1}\right)$, ${\overline{\psi_{\triangle P Q R}}}^{3}\left(v_{1}^{\prime}\right)>v_{1}^{\prime}+1$.

Proof. We assume $\{P, Q, R\}$ are in counterclockwise order without loss of generality. Let $v_{3}, v_{4} \in S^{1}$ be points at infinity that intersect the line $Q R$ and $S^{1}$, and $v_{3}$ is closer to $Q$ than $R$. See Figure 9, Let $v_{1}^{\prime} \in \pi^{-1}\left(v_{1}\right)$. Let $v_{2}^{\prime} \in \pi^{-1}\left(v_{2}\right)$ with $\overline{\psi_{\triangle P Q R}}\left(v_{1}^{\prime}\right)=v_{2}^{\prime}$. Let $v_{3}^{\prime} \in \pi^{-1}\left(v_{3}\right)$ and $v_{4}^{\prime} \in \pi^{-1}\left(v_{4}\right)$ such that $v_{3}^{\prime}<v_{2}^{\prime}<v_{3}^{\prime}+1$ and $\overline{\psi_{\triangle P Q R}}\left(v_{3}^{\prime}\right)=v_{4}^{\prime}$. Because $v_{3}^{\prime}<v_{2}^{\prime}$, we see


Figure 9: $v_{i}(i=1, \ldots, 4)$.
$\overline{\psi_{\triangle P Q R}}\left(\underline{v_{3}^{\prime}}\right)<\overline{\psi_{\triangle P Q R}}\left(v_{2}^{\prime}\right)$. Therefore, we have $\overline{\psi_{\triangle P Q R}}\left(v_{4}^{\prime}\right)<\left(\overline{\psi_{\triangle P Q R}}\right)^{3}\left(v_{1}^{\prime}\right)$. Clearly, $\overline{\psi_{\triangle P Q R}}\left(v_{4}^{\prime}\right)>v_{1}^{\prime}+1$, which implies $\overline{\psi_{\triangle P Q R}}\left(v_{1}^{\prime}\right)^{3}>v_{1}^{\prime}+1$.

Theorem 4.2. Let $\triangle P Q R$ be a triangle in $D$. Then, $\frac{1}{2}>\rho\left(\psi_{\triangle P Q R}\right)>\frac{1}{3}$ if $\delta(P, Q, R)<\Delta^{\prime}(P, Q)$ and $\rho\left(\psi_{\triangle P Q R}\right)=\frac{1}{3}$ if $\delta(P, Q, R) \geq \Delta^{\prime}(P, Q)$.

Proof. First, we suppose $\delta(P, Q, R)<\Delta^{\prime}(P, Q)$. We suppose that there exists $v^{\prime} \in \mathbb{R}$ such that ${\overline{\psi_{\triangle P Q R}}}^{3}\left(v^{\prime}\right) \leq v^{\prime}+1$. From Lemma4.1, there exists $v^{\prime \prime} \in \mathbb{R}$ such that ${\overline{\psi_{\triangle P Q R}}}^{3}\left(v^{\prime \prime}\right)>v^{\prime \prime}+1$. Since $\overline{\psi_{\triangle P Q R}}$ is continuous, there exists $v \in \mathbb{R}$ such that ${\overline{\psi_{\triangle P Q R}}}^{3}(v)=v+1$, which contradicts Theorem 3.1. Therefore, we see that for any $v \in \mathbb{R}{\overline{\psi_{\triangle P Q R}}}^{3}(v)>v+1$. Let $\epsilon:=\min _{v \in \mathbb{R}}\left({\overline{\psi_{\triangle P Q R}}}^{3}(v)-v-1\right)$. Then, we have

$$
\rho\left(\psi_{\triangle P Q R}\right)=\lim _{n \rightarrow \infty} \frac{{\overline{\psi_{\triangle P Q R}}}^{n}(0)}{n} \geq \frac{1}{3}+\frac{\epsilon}{3}>\frac{1}{3} .
$$

Clearly, for any $v \in \mathbb{R}{\overline{\psi_{\triangle P Q R}}}^{2}(v)<1$, which follows $\rho\left(\psi_{\triangle P Q R}\right)<1 / 2$.
Next, we suppose $\delta(P, Q, R) \geq \Delta^{\prime}(P, Q)$. From Theorem 3.1, there exists $v_{1} \in S^{1}$ such that $\psi_{\triangle P Q R}^{3}\left(v_{1}\right)=v_{1}$. Therefore, we have $\rho\left(\psi_{\triangle P Q R}\right)=1 / 3$.

For the case of $\rho\left(\psi_{\triangle P Q R}\right)=1 / 3$, we evaluate the dynamics on $S^{1}$ via the transformation $\psi_{\triangle P Q R}$. For $u, v \in S^{1}$, we denote $\operatorname{arc}(u v)$ by the arc of $S^{1}$ that moves counterclockwise from $u$ to $v$. The following lemma is required.

Lemma 4.3. Let $\triangle P Q R$ be a triangle in $D$ with $\delta(P, Q, R)>\Delta^{\prime}(P, Q)$. Then, there exists $w \in S^{1}$ such that if $w^{\prime} \in \pi^{-1}(w), \overline{\psi \triangle P Q R}^{3}\left(w^{\prime}\right)<w^{\prime}+1$.

Proof. We take a point $A$ in the inner region of $\triangle P Q R$. We consider a similarity transformation $\mathcal{S}_{\lambda}$ for $\lambda \in \mathbb{R}$ defined as for $X \in \mathbb{R}^{2}$

$$
\mathcal{S}_{\lambda}(X):=\lambda(X-A)+A .
$$

It is not difficult to see that

$$
\begin{aligned}
& \lim _{\lambda \rightarrow 0^{+}} \Delta^{\prime}\left(\mathcal{S}_{\lambda}(P), \mathcal{S}_{\lambda}(Q)\right)=\infty \\
& \lim _{\lambda \rightarrow 0^{+}} \delta\left(\mathcal{S}_{\lambda}(P), \mathcal{S}_{\lambda}(Q), \mathcal{S}_{\lambda}(R)\right)=0
\end{aligned}
$$

Because the aforementioned formulas for $\lambda$ are continuous with respect to $\lambda$, there exists $\lambda^{\prime}$ with $0<\lambda^{\prime}<1$ such that

$$
\delta\left(\mathcal{S}_{\lambda^{\prime}}(P), \mathcal{S}_{\lambda^{\prime}}(Q), \mathcal{S}_{\lambda^{\prime}}(R)\right)=\Delta^{\prime}\left(\mathcal{S}_{\lambda}(P), \mathcal{S}_{\lambda}(Q)\right)
$$

We set $P^{\prime}=\mathcal{S}_{\lambda^{\prime}}(P), Q^{\prime}=\mathcal{S}_{\lambda^{\prime}}(Q)$, and $R^{\prime}=\mathcal{S}_{\lambda^{\prime}}(R)$. Then, from Theorem 3.1, there exists $u \in S^{1}$ such that $\psi_{\triangle P^{\prime} Q^{\prime} R^{\prime}}^{3}(u)=u$ and $\psi_{\triangle P^{\prime} Q^{\prime} R^{\prime}}(u) \neq u$. Let $w^{\prime} \in \pi^{-1}(u)$. Then, because triangle $\triangle P^{\prime} Q^{\prime} R^{\prime}$ is included in $\triangle P Q R$, we see ${\overline{\psi_{\triangle P Q R}}}^{3}\left(w^{\prime}\right)<{\overline{\psi_{\triangle P^{\prime} Q^{\prime} R^{\prime}}}}^{3}\left(w^{\prime}\right)=w^{\prime}+1$.

Theorem 4.4. Let $\triangle P Q R$ be a triangle in $D$ and $\rho\left(\psi_{\triangle P Q R}\right)=1 / 3$.
(1) Case of $\delta(P, Q, R)=\Delta^{\prime}(P, Q)$.

There exists $u_{1} \in S^{1}$ such that $\psi_{\triangle P Q R}^{3}\left(u_{1}\right)=u_{1}$. We set $u_{i}:=$ $\psi_{\triangle P Q R}^{i-1}\left(u_{1}\right)$ for $i=2,3$. We set $I_{1}:=\operatorname{arc}\left(u_{3} u_{1}\right) \backslash\left\{u_{3}\right\}$ and $I_{j}:=$ $\operatorname{arc}\left(u_{j-1} u_{j}\right) \backslash\left\{u_{j-1}\right\}$ for $j=2,3$. Then, for $v \in I_{j}$ for $1 \leq j \leq 3$ $\left\{\psi_{\triangle P Q R}^{3 n}(v)\right\}_{n=1,2, \ldots}$ converges to $u_{j}$ as $n \rightarrow \infty$. See Figure 10.
(2) Case of $\delta(P, Q, R)>\Delta^{\prime}(P, Q)$.

There exists $u_{1} \in S^{1}$ such that $\psi_{\triangle P Q R}^{3}\left(u_{1}\right)=u_{1}$ and there exists $w \in$ $S^{1}$ with $v \neq u_{i}(i=1,2,3)$ for which $\lim _{n \rightarrow \infty} \psi_{\triangle P Q R}^{3 n}(w)=u_{1}$. We set $u_{i}:=\psi_{\triangle P Q R}^{i-1}\left(u_{1}\right)$ for $i=2,3$. We set $I_{1}:=\operatorname{arc}\left(u_{3} u_{1}\right) \backslash\left\{u_{3}\right\}$ and $I_{j}:=\operatorname{arc}\left(u_{j-1} u_{j}\right) \backslash\left\{u_{j-1}\right\}$ for $j=2$, 3. There exists $v_{1}\left(\neq u_{2}\right) \in I_{2}$ such that $\psi_{\triangle P Q R}^{3}\left(v_{1}\right)=v_{1}$. We set $v_{i}:=\psi_{\triangle P Q R}^{i-1}\left(v_{1}\right)$ for $i=2,3$. We set $I_{1}^{\prime}:=\operatorname{arc}\left(v_{3} v_{1}\right) \backslash\left\{v_{3}, v_{1}\right\}$ and $I_{j}^{\prime}:=\operatorname{arc}\left(v_{j-1} v_{j}\right) \backslash\left\{v_{j-1}, v_{j}\right\}$ for $j=2,3$. Then, for $v \in I_{j}^{\prime}$ for $1 \leq j \leq 3\left\{\psi_{\triangle P Q R}^{3 n}(v)\right\}_{n=1,2, \ldots}$ converges to $u_{j}$ as $n \rightarrow \infty$. See Figure 11 .


Figure 10: (1) $\psi_{\triangle P Q R}^{3}$.


Figure 11: (2) $\psi_{\triangle P Q R}^{3}$.

Proof. First, we consider (1). We suppose $\delta(P, Q, R)=\Delta^{\prime}(P, Q)$. From Theorem 3.1, there exists $u_{1} \in S^{1}$ such that $\psi_{\triangle P Q R}^{3}\left(u_{1}\right)=u_{1}$ and $\psi_{\triangle P Q R}\left(v_{1}\right) \neq$ $v_{1}$. We define $u_{2}, u_{3}$ and $I_{j} j=1,2,3$ as in the theorem. Because $\psi_{\triangle P Q R}$ is a homeomorphism of $S^{1}$, we see that $\psi_{\triangle P Q R}\left(I_{3}\right)=I_{1}$ and $\psi_{\triangle P Q R}\left(I_{j}\right)=I_{j+1}$ for $j=1,2$. According to Lemma 4.1, there exists $v^{\prime} \in \mathbb{R}$ such that ${\overline{\psi_{\triangle P Q R}}}^{3}\left(v^{\prime}\right)>v^{\prime}+1$. Considering $\pi\left(v^{\prime}\right), \pi\left(\overline{\psi_{\triangle P Q R}}\left(v^{\prime}\right)\right), \pi\left({\overline{\psi_{\triangle P Q R}}}^{2}\left(v^{\prime}\right)\right)$, we see that there exists $v^{\prime \prime} \in \mathbb{R}$ such that $\pi\left(v^{\prime \prime}\right) \in I_{1}$ and

$$
\begin{equation*}
{\overline{\psi_{\triangle P Q R}}}^{3}\left(v^{\prime \prime}\right)>v^{\prime \prime}+1 . \tag{4.1}
\end{equation*}
$$

Let $u \in I_{1}$ and $u \neq u_{1}$. Let $u^{\prime} \in \pi^{-1}(u)$. We suppose that ${\overline{\psi_{\triangle P Q R}}}^{3}\left(u^{\prime}\right) \leq$ $u^{\prime}+1$. Then, from (4.1), there exists $v \in \mathbb{R}$ such that ${\overline{\psi_{\triangle P Q R}}}^{3}(v)=v+$ $1, \pi(v) \in I_{1}$ and $\pi(v) \neq v_{1}$, which contradicts Theorem 3.1. Therefore, ${\overline{\psi_{\triangle P Q R}}}^{3}\left(u^{\prime}\right)>u^{\prime}+1$. Hence, there exists $w \in I_{1} \cup\left\{u_{1}\right\}$ such that

$$
\lim _{n \rightarrow \infty} \psi_{\triangle P Q R}^{3 n}(u)=w
$$

Then, we have $\psi_{\triangle P Q R}^{3 n}(w)=w$, which yields $w=u_{1}$ from Theorem 3.1. Other cases can be proved in the same way.

Next, we consider (2). We suppose $\delta(P, Q, R)>\Delta^{\prime}(P, Q)$. From Theorem 3.1 there exist $b_{1}, c_{1} \in S^{1}$ such that $\psi_{\triangle P Q R}^{3}\left(b_{1}\right)=b_{1}, \psi_{\triangle P Q R}^{3}\left(c_{1}\right)=c_{1}$ and

$$
\cup_{i=1,2,3}\left\{\psi_{\triangle P Q R}^{i-1}\left(b_{1}\right)\right\} \bigcap \cup_{i=1,2,3}\left\{\psi_{\triangle P Q R}^{i-1}\left(c_{1}\right)\right\}=\emptyset .
$$

We set $b_{i}:=\psi_{\triangle P Q R}^{i-1}\left(b_{1}\right)$ for $i=2,3$. We assume that $c_{1} \in \operatorname{arc}\left(b_{1} b_{2}\right)$ without loss of generality. Similarly as in the proof of (1), there exists $v^{\prime} \in \operatorname{arc}\left(b_{1} b_{2}\right)$ such that for $v^{\prime \prime} \in \pi^{-1}\left(v^{\prime}\right){\overline{\psi_{\triangle P Q R}}}^{3}\left(v^{\prime \prime}\right)>v^{\prime \prime}+1$. From Lemma 4.3, there exists $w \in \operatorname{arc}\left(b_{1} b_{2}\right)$ such that for $w^{\prime} \in \pi^{-1}(w),{\overline{\psi_{\triangle P Q R}}}^{3}\left(w^{\prime}\right)<w^{\prime}+1$. First, we suppose $w \in \operatorname{arc}\left(b_{1} c_{1}\right)$. If there exists $v \in \operatorname{int}\left(\operatorname{arc}\left(b_{1} c_{1}\right)\right)$ such that ${\overline{\psi_{\triangle P Q R}}}^{3}\left(v^{\prime}\right) \geq v^{\prime}+1$ for $v^{\prime} \in \pi^{-1}(v)$, then we have the same contradiction as in the proof of (1). As a result, for any $v \in \operatorname{int}\left(\operatorname{arc}\left(b_{1} c_{1}\right)\right)$ we see ${\overline{\psi_{\triangle P Q R}}}^{3}\left(v^{\prime}\right)<$ $v^{\prime}+1$ for $v^{\prime} \in \pi^{-1}(v)$. Then, as in the proof of (1), for every $v \in \operatorname{int}\left(\operatorname{arc}\left(b_{1} c_{1}\right)\right)$

$$
\lim _{n \rightarrow \infty} \psi_{\triangle P Q R}^{3 n}(v)=b_{1}
$$

Similarly, we have for every $v \in \operatorname{int}\left(\operatorname{arc}\left(c_{1} b_{2}\right)\right)$

$$
\lim _{n \rightarrow \infty} \psi_{\triangle P Q R}^{3 n}(v)=b_{2}
$$

Now, we put $u_{1}:=b_{1}$ and $v_{1}:=c_{1}$ and set $u_{2}, u_{3}, v_{2}, v_{3}$, and $I_{j}, I_{j}^{\prime}$ for $j=$ $1,2,3$ as the theorem. Then, we have the theorem. Next, we assume $w \in$ $\operatorname{arc}\left(c_{1} b_{2}\right)$. Similarly, for any $v \in \operatorname{int}\left(\operatorname{arc}\left(c_{1} b_{2}\right)\right)$

$$
\lim _{n \rightarrow \infty} \psi_{\triangle P Q R}^{3 n}(v)=c_{1}
$$

and for every $v \in \operatorname{int}\left(\operatorname{arc}\left(b_{1} c_{1}\right)\right)$

$$
\lim _{n \rightarrow \infty} \psi_{\triangle P Q R}^{3 n}(v)=c_{1}
$$

We put $u_{1}:=c_{1}$ and $v_{1}:=b_{2}$ and set $u_{2}, u_{3}, v_{2}, v_{3}$, and $I_{j}, I_{j}^{\prime}$ for $j=1,2,3$ as the theorem. Then, we have the theorem.

## 5 Application

As we have seen in earlier chapters, the relationship between rotation numbers and the hyperbolic geometric structure when the rotation number is minimized is examined. We would want to talk about whether these properties hold valid for congruent triangle transformations in Euclidean space.

Theorem 5.1. Let $\triangle P Q R$ be a triangle in $D$. Then, there exists a triangle $\triangle P^{\prime} Q^{\prime} R^{\prime}$ in $D$ such that $\triangle P^{\prime} Q^{\prime} R^{\prime}$ is similar to $\triangle P Q R$ and for arbitrary triangle $\triangle A B C \subset D$ which is congruent to $\triangle P^{\prime} Q^{\prime} R^{\prime}, \rho\left(\psi_{\triangle A B C}\right)=\frac{1}{3}$. Here, similarity and congruence are in the sense of Euclidean geometry.
Proof. First, we suppose that $\triangle P Q R$ is an obtuse triangle or right triangle. We may assume $\angle R \geq \pi / 2$. We set $P_{1}=(0,1)$ and $P_{2}=(0,-1)$. It is not difficult to find $P_{3}=\left(x_{1}, x_{2}\right) \in D \cup S^{1}$ such that $\triangle P_{1} P_{2} P_{3}$ is similar to $\triangle P Q R$. We assume $x_{1}>0$ without loss of generality. We take a look at similarity transformation $\mathcal{S}_{\lambda}$ for $\lambda \in \mathbb{R}$ defined as for $X \in \mathbb{R}^{2}$

$$
\mathcal{S}_{\lambda}(X):=\lambda X
$$

We take $K>0$ such that for any $k \geq K$

$$
\begin{equation*}
\frac{4 k\left(k^{2}+1\right)}{(k-1)^{4}}<\frac{x_{1}}{4} \tag{5.1}
\end{equation*}
$$

We take

$$
\begin{equation*}
\epsilon=\min \left\{\frac{x_{1}^{2}}{32}, \frac{1}{4 K}\right\} \tag{5.2}
\end{equation*}
$$

Let $\lambda=1-\epsilon$. Let $P_{i}^{\prime}=\mathcal{S}_{\lambda}\left(P_{i}^{\prime}\right)$ for $i=1,2,3$. Let us demonstrate that $\triangle P_{1}^{\prime} P_{2}^{\prime} P_{3}^{\prime}$ is as specified in the theorem. We note that when considering a congruent transformation to $\triangle P_{1}^{\prime} P_{2}^{\prime} P_{3}^{\prime}$ in $D$, we just need to consider translation, not rotation or symmetry transformations by symmetry. For $\tau_{1}, \tau_{2} \in \mathbb{R}^{2}$ we define the translation $\mathcal{P}_{\tau_{1}, \tau_{2}}$ as for $(x, y) \in \mathbb{R}^{2} \mathcal{P}_{\tau_{1}, \tau_{2}}(x, y):=\left(x+\tau_{1}, y+\tau_{2}\right)$. We define the set $U \subset D$ as

$$
U:=\left\{\left(\tau_{1}, \tau_{2}\right) \in \mathbb{R}^{2} \mid \mathcal{P}_{\tau_{1}, \tau_{2}}\left(\triangle P_{1}^{\prime} P_{2}^{\prime} P_{3}^{\prime}\right) \subset D\right\}
$$

Then, we see

$$
\begin{equation*}
U \subset\left\{\left(\tau_{1}, \tau_{2}\right) \in \mathbb{R}^{2} \mid \mathcal{P}_{\tau_{1}, \tau_{2}}\left(P_{1}^{\prime} P_{2}^{\prime}\right) \subset D\right\} \subset(-\sqrt{2 \epsilon}, \sqrt{2 \epsilon}) \times(-\epsilon, \epsilon) \tag{5.3}
\end{equation*}
$$

Let $\left(\tau_{1}^{\prime}, \tau_{2}^{\prime}\right) \in U$. We set $V_{i}:=\mathcal{P}_{\tau_{1}^{\prime}, \tau_{2}^{\prime}}\left(P_{i}\right)$ for $i=1,2,3$. From the facts that $\left|\tau_{2}^{\prime}\right|<\epsilon,\left(\tau_{1}^{\prime}, \tau_{2}^{\prime}\right) \in U$, and (5.2), we get

$$
\left.\begin{array}{l}
d^{\prime}\left(V_{1}, V_{2}\right) \\
=\frac{1}{2} \log \left(\frac{\left(1-\epsilon+\tau_{2}^{\prime}+\sqrt{1-\left(\tau_{1}^{\prime}\right)^{2}}\right)\left(1-\epsilon-\tau_{2}^{\prime}+\sqrt{1-\left(\tau_{1}^{\prime}\right)^{2}}\right)}{\left(-1+\epsilon-\tau_{2}^{\prime}+\sqrt{1-\left(\tau_{1}^{\prime}\right)^{2}}\right)\left(-1+\epsilon+\tau_{2}^{\prime}+\sqrt{1-\left(\tau_{1}^{\prime}\right)^{2}}\right)}\right.
\end{array}\right)
$$

We set $k^{\prime}:=e^{d^{\prime}\left(V_{1}, V_{2}\right)}$. Then, we have

$$
\begin{equation*}
k^{\prime}>\frac{1}{4 \epsilon} . \tag{5.4}
\end{equation*}
$$

Let $E$ be the ellipse defined for $V_{1}, V_{2}$ in Theorem 3.3; i.e., if $A, B$, and $C$ are as defined in Theorem 3.3, we have

$$
E=\left\{(x, y) \left\lvert\, \frac{(x-C)^{2}}{A^{2}}+\frac{y^{2}}{B^{2}}=1\right.\right\}
$$

In particular, $A$ is as follows:

$$
\begin{equation*}
A:=\frac{2 k^{\prime}\left(k^{\prime 2}+1\right)\left(1-\tau_{1}^{\prime 2}\right)}{\left(k^{\prime 2}-2 k^{\prime} \tau_{1}^{\prime}+1\right)\left(k^{\prime 2}+2 k^{\prime} \tau_{1}^{\prime}+1\right)} . \tag{5.5}
\end{equation*}
$$

Consider $\Phi$ to be the inner region enclosed by $E$. Since $\left(\tau_{1}^{\prime}, 0\right)$ is in $\Phi, \Phi$ is included in the set $\left\{(x, y)\left|\left|x-\tau_{1}^{\prime}\right|<2 A\right\}\right.$ denoted by $\Phi^{\prime}$. Let us demonstrate $V_{3} \notin \Phi^{\prime}$. From (5.1), (5.2), (5.4), and (5.5) we have

$$
\begin{equation*}
2 A<\frac{4 k^{\prime}\left(k^{\prime 2}+1\right)}{\left(k^{\prime}-1\right)^{4}}<\frac{x_{1}}{4} . \tag{5.6}
\end{equation*}
$$

We recall $V_{3}=\mathcal{P}_{\tau_{1}^{\prime}, \tau_{2}^{\prime}}\left(P_{3}^{\prime}\right)=\left((1-\epsilon) x_{1}+\tau_{1}^{\prime},(1-\epsilon) x_{2}+\tau_{2}^{\prime}\right)$. From (5.2) and (5.6), we have

$$
(1-\epsilon) x_{1}>\frac{31}{32} x_{1}>2 A
$$

which implies $V_{3} \notin \Phi^{\prime}$. Therefore, from Theorem 3.3, we see $\rho\left(\psi_{\Delta V_{1} V_{2} V_{3}}\right)=\frac{1}{3}$. Next, we assume that $\triangle P Q R$ is an acute triangle. Then, there is $\triangle P_{1} P_{2} P_{3}$, which is inscribed in $S^{1}$ and is similar to $\triangle P Q R$. We can assume that $P_{1} P_{2}$ is parallel to the $y$-axis. We can provide comparable proof in this case as we did in the previous ones. We leave the proof to the reader.

If $\triangle P Q R$ is similar to $\triangle P^{\prime} Q^{\prime} R^{\prime}$, we denote by $\triangle P Q R \sim \triangle P^{\prime} Q^{\prime} R^{\prime}$. For a triangle $\triangle P Q R$ in $\mathbb{R}^{2}$, we define $\kappa(\triangle P Q R)$ as its circumradius if $\triangle P Q R$ is an acute triangle, and $\kappa(\triangle P Q R)$ as half of the length of the largest side of $\triangle P Q R$ if $\triangle P Q R$ is an obtuse triangle or right triangle.

For a triangle $\triangle P Q R$ in $\mathbb{R}^{2}$, we define $\mu(\triangle P Q R)$ as the infimum of $\kappa\left(\triangle P^{\prime} Q^{\prime} R^{\prime}\right)$ such that
(1) $\triangle P^{\prime} Q^{\prime} R^{\prime} \subset D$,
(2) $\triangle P^{\prime} Q^{\prime} R^{\prime} \sim \triangle P Q R$,
(3) for any $\triangle P^{\prime \prime} Q^{\prime \prime} R^{\prime \prime} \subset D$ which is congruent to $\triangle P^{\prime} Q^{\prime} R^{\prime}, \rho\left(\psi_{\triangle P^{\prime \prime} Q^{\prime \prime} R^{\prime \prime}}\right)=$ $\frac{1}{3}$.
$\mu(\triangle P Q R)$ is well defined from Theorem 5.1. Furthermore, we have
Proposition 5.2. Let $\triangle P Q R$ be a triangle in $\mathbb{R}^{2}$. Then, $0<\mu(\triangle P Q R)<$ 1.

Proof. For any $\triangle P^{\prime} Q^{\prime} R^{\prime} \subset D$, we see $\kappa\left(\triangle P^{\prime} Q^{\prime} R^{\prime}\right)<1$, which implies $\mu(\triangle P Q R)<1$. Let $\triangle P^{\prime} Q^{\prime} R^{\prime} \subset D$ be similar to $\triangle P Q R$. For $\lambda \in \mathbb{R}$, we define the similarity transformation $\mathcal{S}_{\lambda}$ as for $X \in \mathbb{R}^{2}$

$$
\mathcal{S}_{\lambda}(X):=\lambda X
$$

It is not difficult to see that

$$
\begin{aligned}
& \lim _{\lambda \rightarrow 0} \delta\left(\mathcal{S}_{\lambda}\left(P^{\prime}\right), \mathcal{S}_{\lambda}\left(Q^{\prime}\right), \mathcal{S}_{\lambda}\left(R^{\prime}\right)\right)=0, \\
& \lim _{\lambda \rightarrow 0} \Delta^{\prime}\left(\mathcal{S}_{\lambda}\left(P^{\prime}\right), \mathcal{S}_{\lambda}\left(Q^{\prime}\right)\right)=\infty
\end{aligned}
$$

Therefore, from Theorem 3.1, there exists $\lambda^{\prime}>0$ such that for every $\lambda$ with $0<\lambda<\lambda^{\prime} \rho\left(\psi_{\left.\left.\left.\left.\triangle \mathcal{S}_{\lambda}\left(P^{\prime}\right)\right) \mathcal{S}_{\lambda}\left(Q^{\prime}\right)\right) \mathcal{S}_{\lambda}\left(R^{\prime}\right)\right)\right)}>\frac{1}{3}\right.$, which follows $0<\mu(\triangle P Q R)$.


Figure 12: Points in the proof of Proposition 5.3,

We have not yet established the method to compute $\mu(\triangle P Q R)$. We give an example.

Proposition 5.3. Let $\triangle P Q R$ be an equilateral triangle in $\mathbb{R}^{2}$. Then, $\mu(\triangle P Q R)=$ $\frac{1}{2}$.

Proof. We put $P=\left(-\frac{1}{4}, \frac{\sqrt{3}}{4}\right), Q=\left(-\frac{1}{4},-\frac{\sqrt{3}}{4}\right), R=\left(\frac{1}{2}, 0\right)$, and $S=\left(-\frac{1}{4}, 0\right)$. Then, $\triangle P Q R$ is an equilateral triangle and $\rho\left(\psi_{\triangle P Q R}\right)=\frac{1}{3}$ as seen in Example 3.1. First, we consider $\mathcal{P}_{\tau_{1}, 0}(\triangle P Q R)$ for $\tau_{1} \in \mathbb{R}$. We note that $\mathcal{P}_{\tau_{1}, 0}(\triangle P Q R) \subset D$ implies $-\frac{\sqrt{13}-1}{4}<\tau_{1}<\frac{1}{2}$. We set $P^{\prime}=\mathcal{P}_{\tau_{1}, 0}(P)$, $Q^{\prime}=\mathcal{Q}_{\tau_{1}, 0}(Q), R^{\prime}=\mathcal{P}_{\tau_{1}, 0}(R)$, and $S^{\prime}=\mathcal{P}_{\tau_{1}, 0}(S)$ for $-\frac{\sqrt{13}-1}{4}<\tau_{1}<\frac{1}{2}$ (see Figure (12). Let $u=-\frac{1}{4}+\tau_{1}$. Then, $P^{\prime}=\left(u, \frac{\sqrt{3}}{4}\right), Q^{\prime}=\left(u,-\frac{\sqrt{3}}{4}\right)$, $R^{\prime}=\left(u+\frac{3}{4}, 0\right)$, and $S^{\prime}=(u, 0)$. We note $-\frac{\sqrt{13}}{4}<u<\frac{1}{4}$. Furthermore, we put $v_{1}=\left(u, \sqrt{1-u^{2}}\right), v_{2}=\left(u,-\sqrt{1-u^{2}}\right), v_{3}=(1,0)$, and $v_{4}=(-1,0)$. Then, we have

$$
\begin{aligned}
& d^{\prime}\left(P^{\prime}, Q^{\prime}\right)=\log \frac{\sqrt{1-u^{2}}+\sqrt{3} / 4}{\sqrt{1-u^{2}}-\sqrt{3} / 4} \\
& \Delta^{\prime}\left(P^{\prime}, Q^{\prime}\right)=\log \frac{e^{d^{\prime}(P, Q)}+1}{e^{d^{\prime}(P, Q)}-1} \\
& \quad=\log \frac{\sqrt{1-u^{2}}+\sqrt{3} / 4}{\sqrt{1-u^{2}}-\sqrt{3} / 4}+1 \\
& \frac{\sqrt{1-u^{2}}+\sqrt{3} / 4}{\sqrt{1-u^{2}}-\sqrt{3} / 4}-1
\end{aligned}=\log \frac{4 \sqrt{1-u^{2}}}{\sqrt{3}} .
$$

Since $\delta\left(P^{\prime}, Q^{\prime}, R^{\prime}\right)=d^{\prime}\left(R^{\prime}, S^{\prime}\right)$, we have

$$
\delta\left(P^{\prime}, Q^{\prime}, R^{\prime}\right)=\frac{1}{2} \log \frac{(u+7 / 4) \cdot(1-u)}{(1 / 4-u) \cdot(u+1)}
$$

We see

$$
\begin{aligned}
& \frac{(u+7 / 4) \cdot(1-u)}{(1 / 4-u) \cdot(u+1)}-\left(\frac{4 \sqrt{1-u^{2}}}{\sqrt{3}}\right)^{2} \\
& =\frac{(1-u)(4 u+1)^{2}(4 u+5)}{3(u+1)(1-4 u)} \geq 0
\end{aligned}
$$

where the equality holds if and only if $u=-\frac{1}{4}$. Therefore, we see

$$
\begin{equation*}
\delta\left(P^{\prime}, Q^{\prime}, R^{\prime}\right) \geq \Delta^{\prime}\left(P^{\prime}, Q^{\prime}\right) \tag{5.7}
\end{equation*}
$$

where the equality holds if and only if $\triangle P^{\prime} Q^{\prime} R^{\prime}=\triangle P Q R$. Therefore, $\rho\left(\psi_{\triangle P^{\prime} Q^{\prime} R^{\prime}}\right)=\frac{1}{3}$. Next, let us consider a translation parallel to the $y$-axis; i.e., we consider $\mathcal{P}^{\prime}{ }_{0, \tau_{2}}\left(\triangle P^{\prime} Q^{\prime} R^{\prime}\right)$ for $\tau_{2} \in \mathbb{R}$. We note that we only consider the case of $\tau_{2}>0$ by symmetry. Therefore, we suppose $\tau_{2}>0$ and we have $0<\tau_{2}<\sqrt{1-u^{2}}-\frac{\sqrt{3}}{4}$. We set $P^{\prime \prime}=\mathcal{P}_{0, \tau_{2}}\left(P^{\prime}\right), Q^{\prime \prime}=\mathcal{P}_{0, \tau_{2}}\left(Q^{\prime}\right)$, $R^{\prime \prime}=\mathcal{P}_{0, \tau_{2}}\left(R^{\prime}\right)$, and $S^{\prime \prime}=\mathcal{P}_{0, \tau_{2}}\left(S^{\prime}\right)$.

Then, we have

$$
\begin{equation*}
d^{\prime}\left(P^{\prime \prime}, Q^{\prime \prime}\right)=\frac{1}{2} \log \frac{\left(\sqrt{1-u^{2}}+\sqrt{3} / 4+\tau_{2}\right)\left(\sqrt{1-u^{2}}+\sqrt{3} / 4-\tau_{2}\right)}{\left(\sqrt{1-u^{2}}-\sqrt{3} / 4+\tau_{2}\right)\left(\sqrt{1-u^{2}}-\sqrt{3} / 4-\tau_{2}\right)} \tag{5.8}
\end{equation*}
$$

In general, if $A>B>x>0$ for $A, B, x \in \mathbb{R}$, we have $\frac{(A+x)(A-x)}{(B+x)(B-x)}>\frac{A^{2}}{B^{2}}$. Therefore, from (5.8), we have $d^{\prime}\left(P^{\prime \prime}, Q^{\prime \prime}\right)>d^{\prime}\left(P^{\prime}, Q^{\prime}\right)$, which implies

$$
\begin{equation*}
\Delta^{\prime}\left(P^{\prime \prime}, Q^{\prime \prime}\right)<\Delta^{\prime}\left(P^{\prime}, Q^{\prime}\right) \tag{5.9}
\end{equation*}
$$

Let $E$ (resp., $E^{\prime}$ ) be the ellipse related to $P^{\prime}, Q^{\prime}\left(\right.$ resp., $\left.P^{\prime \prime}, Q^{\prime \prime}\right)$ in Theorem 3.3. Let $\Phi$ (resp., $\Phi^{\prime}$ ) denote the inner region bounded by $E$ (resp., $E^{\prime}$ ). We can see from (5.7) that $R^{\prime}$ is not included in $\Phi$. Because $E$ is an ellipse with its center in the $x$-axis and one of its axes parallel to the $y$-axis, we obtain $\Phi \subset\left\{(x, y) \in \mathbb{R}^{2} \left\lvert\, x<u+\frac{3}{4}\right.\right\}$. From the fact that line $P^{\prime} Q^{\prime}$ is equal to line $P^{\prime \prime} Q^{\prime \prime}$ and (5.9), we have $\Phi^{\prime} \subset \Phi$. Therefore, $R^{\prime \prime}=\left(u+\frac{3}{4}, \tau_{2}\right)$ is not included in $\Phi^{\prime}$. Therefore, $\rho\left(\psi_{\triangle P^{\prime \prime} Q^{\prime \prime} R^{\prime \prime}}\right)=\frac{1}{3}$. Thus, we show that the rotation number of $\psi_{\triangle P Q R}$ is invariant under translation, which means that it is invariant under congruent transformations by symmetry. It is observable that if we reduce $\triangle P Q R$ even slightly closer to the origin, the related rotation number is $>\frac{1}{3}$. As a result, the proof is complete.

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