

Representing Topological Structures Using Cell-Chains*

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Abstract

A new topological representation of surfaces in higher dimensions, “cell-chains” is developed. The representation is a generalization of Brisson’s cell-tuple data structure. Cell-chains are identical to cell-tuples when there are no degeneracies: cells or simplices with identified vertices. The proof of correctness is based on axioms true for maps, such as those in Brisson’s cell-tuple representation. A critical new condition (axiom) is added to those of Lienhardt’s n-G-maps to give “cell-maps”. We show that cell-maps and cell-chains characterize the same topological representations.

Keywords: computational topology, cell complex, cell tuple, cell chain

1 Extended Abstract

The goal of a topological data structure is to maintain the cells and incidence relationship between these cells in such a way that topological and other information can be stored and retrieved correctly and efficiently. This problem has been cited in a 1999 white paper by Bern et al. as an important open area of research in computational topology.

The most popular data structure is the cell-complex of Brisson in papers from 1989 and 1993. In this structure, topological information is stored via **cell-tuples**, which are maximal paths in the **incidence graph** or **incidence poset**. This data structure yields extremely efficient and elegant implementations for many operations on topological arrangements, and it is widely used in practice.

The two main limitations of Brisson’s cell-complex is that it can only represent a very regular class of structures and that the test for membership in this class is undecidable in higher dimensions. To address the first limitation, we propose an extension to cell-complexes that allows for representation of an **incidence multi-graph**, thus allowing various types of degeneracies. We operate using **cell-chains**, which are maximal paths in the multi-graph.

The second limitation of Brisson’s cell complexes is that he requires that the cell-tuple representation comes from a manifold, a space locally homeomorphic to \mathbb{R}^d , for a d -dimensional structure. It is known that deciding if a topological structure is locally homeomorphic to \mathbb{R}^d is undecidable for $d \geq 6$. By the work of Markov from 1958 and Volodin et al. 1974, it is open for $d = 5$, and recently shown to be in *NP* for $d = 4$ by Schleimer. To address the undecidability problem inherent in

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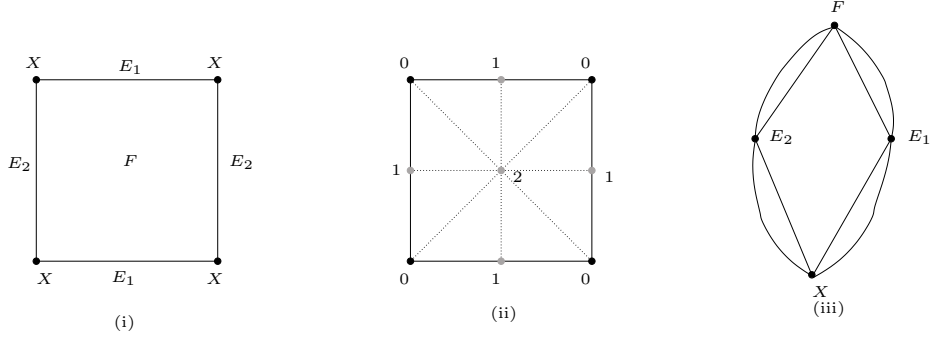


Figure 1: (i) A simple torus formed with a single face, two edges, and one vertex. (ii) The barycentric subdivision of this torus. The barycenter of each cell is inserted and labeled with the dimension of the cell. The result is triangulation using numbered simplices. (iii) The incidence multi-graph for the torus. Edges connects neighboring cells whose dimension differs by 1. In this example the face is doubly connected to each edge and each edge is doubly connected to the vertex

Brisson’s cell complexes, we do not require our representation be locally homeomorphic to \mathbb{R}^d . We only require our representation to satisfy testable conditions that Brisson’s representations satisfy. These conditions are true for many degenerate structures also. The combinatorial axioms given can be verified in $O(nd)$ where n is the input size and d is the dimension. Our axiomatic approach follows those developed for algebraic representations.

The benefit of algebraic representations is that they have axiomatic definitions for the combinatorial structure of the topology structure that is being represented. The vast majority of algebraic representations (some in hindsight) use permutation groups that act on the simplices of the barycentric subdivision or barycentric complex. The barycentric complex is constructed bottom-up by “gluing” together numbered simplices. Axiomatic gluing rules are given so that the complex will be well-formed (in terms of whatever algebraic structure is involved). Throughout the paper, we call a set of gluing rules a map. The most popular data structures of this form are n -G-maps, due to Lienhardt, which uses a weak set of axioms, thus constructing a very large class of topological structures.

We propose an additional axiom to those of Lienhardt, the **Orthogonality Axiom**, that restricts the possible combinatorial structures, but still allows a rich class of topological structures.

The central result of this work is **Cell-Chain Representation Theorem**, wherein we show that our cell-chain extension to Brisson’s cell-complex is precisely equivalent to our axiomatic restriction of Lienhardt’s n -G-maps. We also give a nearly linear time test for our axioms.

This middle ground allows for an intuitive data structure that still represents a large group of topologies, thus combining the benefits of both major approaches to topological data structures. We have implemented two-dimensional implementation of cell-chains within the TUMBLE software package at CMU.

An interesting simple example of an application of cell-chains is the representation of the torus shown in Figure 1.