

Unfolding Well-Separated Orthotrees

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1 Introduction

Because of the difficulty of the long-standing open problem of deciding whether every convex polyhedron can be edge-unfolded [DO05], attention has turned to various specializations or alterations of the original problem. To *edge-unfold* the surface of a polyhedron is to cut a collection of edges so that the surface may be unfolded to a planar, single-piece *net*. One line of investigation, started in [BDD⁺98], focuses on *orthogonal polyhedra*—those whose faces meet at angles that are multiples of 90° . Although not every orthogonal polyhedron has an edge unfolding, in [BDD⁺98] it is shown that *orthostacks* have an unfolding with some cuts interior to faces (a *general unfolding*), and *orthotubes* have an edge unfolding. Subsequent work [DM04] established that a subclass of orthostacks can be edge-unfolded. The work we report here is closest to that on orthotubes, which are polyhedra made by gluing boxes face-to-face such that the dual graph (each node a box, arcs corresponding to glued faces) is a path or cycle. An *orthotree* is a similar polyhedron, except with the condition that the dual graph be a tree. We use the convention that each box edge is an edge of the polyhedron available for cutting (i.e., edges between coplanar faces are not “erased”). Thus, our orthotrees already include the *vertex grid* of edges formed by intersecting the polyhedron with coordinate planes through every box vertex. The edges of the vertex grid offer more options for edge-unfolding; see [DIL04] and [DFO05].

Our main result is that a subclass of orthotrees, “well-separated” orthotrees, have an edge unfolding. The algorithm is naturally recursive on the tree structure, and we believe it shows promise for extension.

2 Definitions

An *orthotree* O is a polyhedron made out of boxes that are glued face-to-face such that the dual graph $G = (V, E)$ of O

has degree k if its dual vertex has degree k in G . A box b_i is a *leaf* if it has degree one; b_i is a *connector* if it has degree two, and its two neighbors are glued to opposite faces of b_i ; otherwise, b_i is a *junction*. An orthotree is *well-separated* if no neighbor of a junction is another junction, i.e., all neighbors of junctions are either leaves or connectors. See Fig. 1.

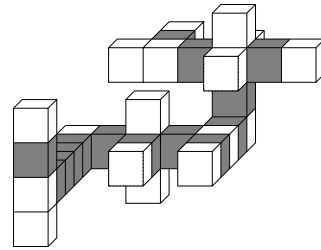


Figure 1: A well-separated orthotree; connectors are grey.

In the remainder of this paper, an edge will refer to one of the 12 edges of a box in an orthotree. Any box of degree k has $6 - k$ exposed faces. In this paper we show that well-separated orthotrees can be edge-unfolded without overlap. We do allow, however, non-neighboring faces to be placed side by side.

The two faces of a box b_i in O perpendicular to the x -axis are denoted by x_i^+ and x_i^- . The face x_i^+ is the face whose outward normal is the positive x -axis. Similarly we define the faces y_i^+ , y_i^- , z_i^+ and z_i^- . The four edges of x_i^+ and x_i^- are labeled f , b , u and d , for front, back, up and down. The edges of y_i^+ and y_i^- are labeled f , b , w and e , with w and e denoting west and east. Faces z_i^+ and z_i^- contain labels u , d , w and e . So an edge has multiple labels. Fig. 2 illustrates this notation and an unfolding of a single box orthotree.

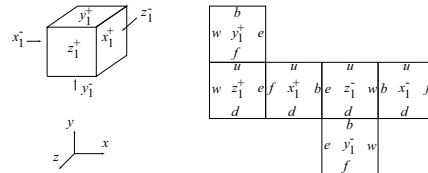


Figure 2: Notation and unfolding of a single box.

3 Unfolding Techniques

Select any leaf box as the root of O , a well-separated orthotree. Our main result is stated in Theorem 1.

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Theorem 1 For any connector or leaf b_0 in O , the subtree rooted at b_0 can be unfolded without overlap, and in two different ways, as illustrated in Fig. 3.

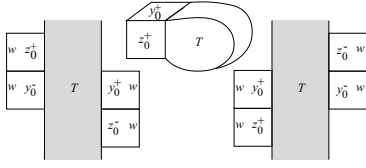


Figure 3: The unfolding of T fits within the shaded area and attaches to a z_0 or y_0 face on either side. Face x_0^- is not shown.

Proof sketch: The proof is by induction on the number of boxes in the subtree T_0 rooted at b_0 . The base case corresponds to a single box subtree for which the two unfoldings can be easily derived.

The induction assumption is that Theorem 1 holds for subtrees with fewer than h nodes, giving us two unfoldings. Observe that by reversing these two unfoldings, we get unfoldings starting from the remaining two adjacent face pairs (e.g. pairs y_0^+, z_0^- and z_0^-, y_0^- in Fig 3). To prove the inductive step, consider a subtree T_0 with h nodes rooted at connector b_0 . W.l.o.g, assume that $T_0 \setminus \{b_0\}$ attaches to x_0^+ . Let b_1 be the box in T_0 glued to x_0^+ . We distinguish six cases, depending on the degree of b_1 . Here we only have space to discuss the cases when b_1 is a junction of degree 5 and 6. For any i , let T_i be the subtree rooted at b_i .

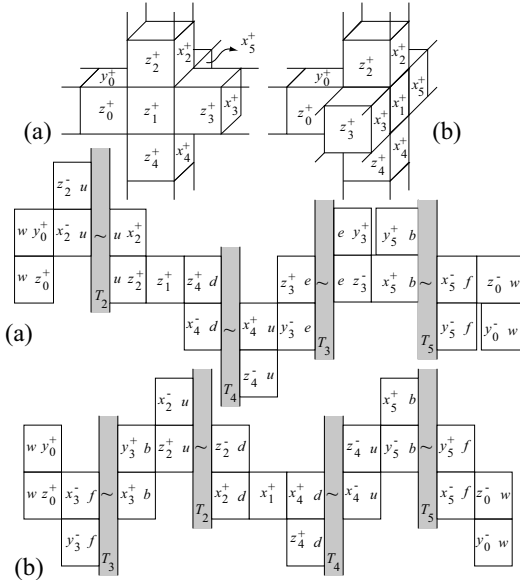


Figure 4: Box b_1 is a junction of degree 5.

Case 1: b_1 is a junction of degree 5. There are only two distinct cases for a degree 5 junction; see Figs. 4a and 4b. We can assume w.l.o.g. that the junction is oriented as shown, since we will provide unfoldings

starting from each pair of adjacent faces of b_0 , i.e. the two inductive step unfoldings and their reverses. The first case is shown in Fig. 4a: starting at y_0^+ , unfold T_2 first, then move across z_1^+ to unfold T_4, T_3 , and T_5 , and finally back to b_0 . Because the orthotree is well-separated, T_2, \dots, T_5 are rooted at connectors or leaves and can be recursively handled. The second unfolding corresponding to Fig. 3a is equally easy to find. In Fig. 4b, the unfolding order is T_3, T_2, x_1^+ to get us to T_4, T_5 , and back to b_0 .

Case 2: b_1 is a junction of degree 6. One unfolding is shown in Fig. 5. Due to symmetry, the second unfolding is a horizontal mirror image of this one.

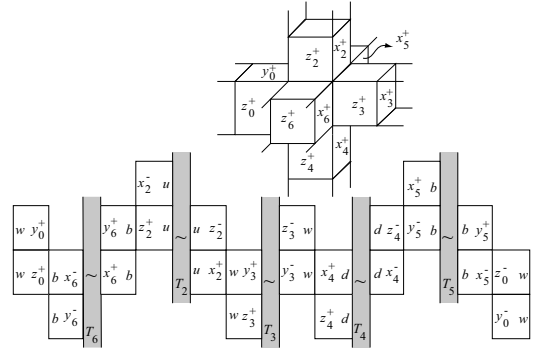


Figure 5: Interior box b_1 is a junction of degree 6.

We note that nothing in our algorithm depends on the boxes being cubes. The obvious open problem is to remove the well-separated assumption.

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