Enumerating Planar Minimally Rigid Graphs

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Motivated by the work of Kawamoto et al. [5], who first suggested the use of graph enumeration techniques as an engineering tool for finding an optimum mechanism design, we give an algorithm for **enumerating all the** *planar Laman graphs* **embedded on a given generic set** p **of** n **points.** Our algorithm is based on the *Reverse search* paradigm of Avis and Fukuda [1]. In particular, we obtain that the set of all planar Laman graphs on a given point set is connected by flips which remove an edge and then restore the Laman property with the addition of a non-crossing edge.

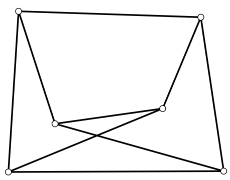

Figure 1: A non-planar Laman graph.

A graph on n vertices is a *Laman graph* if it has exactly $2n-3$ edges and every subset of $n' < n$ vertices spans at most $2n'-3$ edges. A classical result in Rigidity Theory [3], due to Laman, states that the underlying graphs of generic minimally rigid bar-and-joint frameworks in dimension 2 are exactly the Laman graphs. A *planar* minimally rigid framework on a given generic two-dimensional point set p is a Laman graph embedded on p in such a way that non-adjacent edges do not cross. Not all Laman graphs are planar: the simplest example is $K_{3,3}$ (Fig. 1). Not all planar Laman graphs admit non-crossing embeddings on a given point set. An example (Fig. 2) is the graph of the triangular prism, which cannot be embedded with non-crossing edges if, for instance, the point set is in convex position.

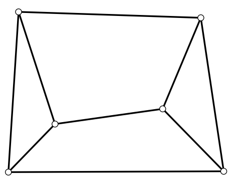

Figure 2: The graph of the triangular prism.

A special subclass of embedded planar Laman graphs are the *pointed pseudo-triangulations* [6]. For them, Bereg [2] has proposed a very efficient enumeration algorithm. Very little is known about the set of *all* the planar Laman graphs on a given point set. While every Laman graph has an embedding as a pointed pseudo-triangulation [4], it is *not true* that *for a given point set*, the *embedded* planar Laman graphs coincide with the pointed pseudo-triangulations.

The Laman graphs on n vertices form the set of *bases* of the *rigidity matroid* in dimension 2, see [3]. The matroid is defined on the set of edges E, and a basis is a collection of edges. All the bases have the same size (in the case of Laman graphs, they all have $2n-3$ elements). Two bases are related via the *basis exchange* operation, an abstraction of *pivoting* from linear programming. In our context we will call this operation a *flip* between two Laman graphs. Two

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Laman graphs G_1 and G_2 are connected by a flip if their edge sets agree on all but one position. The flip is given by the pair of edges (e_1, e_2) in their difference, $e_1 \in G_1 \setminus G_2$, $e_2 \in G_2 \setminus G_1$. Using flips, we can define a graph whose nodes are *all* the Laman graphs on n vertices, and whose edges correspond to flips. It is well-known that the graph whose nodes are the bases of a matroid connected via flips, is connected. However, the *planar* Laman graphs form just a subset of the bases: a priori, this set may not even be connected.

The Search Tree. The main (conceptual) structure required by Reverse Search is a *search tree* on the set of all the planar Laman graphs of a given point set. We choose the root of the search tree to be a *greedy pseudo-triangulation* corresponding to a fixed direction. Then we define a *parent* for every non-root Laman graph using the slope ordering of the edges of the graph. To show that the resulting directed graph structure is acyclic, we associate an *index* to every planar Laman framework, which captures a certain distance from the corresponding node to the root. The lexicographic ordering on these indices attains a minimum at the root, and the parent of a non-root node is smaller than the node. This gives a forest structure. To prove that it is connected and thus a Search *Tree*, we must show the existence of a parent for every non-root node.

The following two theorems imply that the parent edges induce a tree structure, and hence the correctness of our Reverse-Search-based algorithm.

Theorem 1. Every non-root planar Laman graph has a parent.

Theorem 2. For every planar Laman graph different from the root, the index of the *parent* is smaller than the index of the node.

Algorithm. The search algorithm will traverse the tree (e.g. in prefix order), starting from the root and following the parent edges in reverse. We use a the *lexicographic rule* for generating all the children of a node via flips that first remove an edge and then add one to restore the Laman property. For each candidate pair of removed-added edges, the algorithm has to check that the graph obtained by applying the flip has the current node as *parent*. Otherwise, the child node is not (yet) generated. Note that the algorithm doesn't use any additional storage, besides what is needed for the current graph.

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