Optimal Shape of a Blob

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This paper presents the solution to the following optimization problem: What is the shape of the region that minimizes the average L_p distance between all pairs of points if the area of this region is held fixed? (The L_p-distance between two points $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ in $\Re^2$ is $(|x_1 - y_1|^p + |x_2 - y_2|^p)^{1/p}$.)

We use variational methods to determine the shape of the region that minimizes the average L_p distance when the area of this region is held fixed, for $p \geq 1$. Specifically, we characterize the boundary curve $h(x)$ of the optimal region and show that this boundary curve satisfies an integral equation. This integral equation is new and is the principal result of this paper.

This paper generalizes two earlier studies. Karp, McKellar, and Wong [2] considered the two special extreme cases $p = 1$ and $p = \infty$. For these cases, they determined a differential equation that describes the boundary of the optimal region that minimizes the average distance between all points in the region. Bender et al. [1] studied the dimensionless average pairwise distance $D[h]$ that characterizes the degree of optimality of the region for $p = 1$ and they used variational methods to minimize the value of this functional $D[h]$. This variational approach leads directly to the nonlinear differential equation that describes the boundary curve. Moreover, in [1] the numerical value of $D[h]$ was computed.

Here we extend the variational methods introduced in Ref. [1] to the general case $p \geq 1$. For this situation the boundary curve satisfies an integral equation. We then examine three special cases: For $p = 2$ the integral equation reduces to a differential equation, whose solution, as one would expect, is a circle. For $p = 1$ and for $p = \infty$ the integral equation reduces to the second-order nonlinear differential equations that were studied previously in Refs. [1, 2].

In this talk we will also review the results of Ref. [1].

References

[1] C. M. Bender, M. A. Bender, E. Demaine, and S. Fekete. What is the optimal shape of a city? *Journal of Physics A: Mathematical and General*, 37:147–159, 2004.

[2] R. M. Karp, A. C. McKellar, and C. K. Wong. Near-optimal solutions to a 2-dimensional placement problem. *SIAM Journal on Computing*, 4:271–286, 1975.