

Combinatorial and Algorithmic Applications of the Borsuk-Ulam Theorem

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1 Introduction and Summary

The Borsuk-Ulam theorem states that if $f : S^n \rightarrow R^n$ is a continuous mapping from the unit sphere in R^{n+1} into R^n , there is a point $x \in S$ where $f(x) = f(-x)$; i.e., some pair of antipodal points has the same image. The recent book of Matoušek [21] is devoted to explaining this theorem, its background, and some of its many consequences in algebraic topology, algebraic geometry, and combinatorics. Borsuk-Ulam is a great theorem because it has several different equivalent versions, many different proofs, many extensions and generalizations, and many interesting applications.

One familiar consequence is the ham-sandwich theorem (given d finite continuous measures on R^d , there exist a hyperplane that simultaneously bisects them), along with some of its extensions and generalizations to partitioning continuous measures [2], [6], [7], [8], [10]. In many cases there are discrete versions of these results, and it is interesting and instructive to find direct, combinatorial proofs. In addition, there are algorithmic issues about the computational complexity of finding the asserted combinatorial object. For example Lo et. al. [20] gave a direct combinatorial proof of the discrete ham-sandwich theorem and described algorithms to compute ham-sandwich cuts for point sets. Various generalizations and extensions were considered in [1], [2], [3], [4], [5], [9], [10], [11], [12], [17], [18], [19], [23], and [25].

A recent interesting example extends a result of Bárány and Matoušek [7], who combined Borsuk-Ulam with equivariant topology to show that three finite, continuous measures on R^2 can be equipartitioned by a *2-fan*, the partition of R^2 determined by two half-lines incident at a point. Bereg [10] proved a discrete version of this result, and in a stronger form. He showed that given $2r$ red points, $2b$ blue points, and $2g$ green points in general position in R^2 , and also given a line ℓ , there is a point $P \in \ell$ and rays ρ_1 and ρ_2 incident at P (a *2-fan*), that equipartition each set of points: i.e., r reds, b blues, g greens lie in the open wedge defined by the rays. He also described a beautiful algorithm to find such a partitioning. We show his algorithm is nearly optimal by proving the following result.

Lemma 1 *Let S be a given set of points in R^2 , a of them red, b of them blue, and c of them green. For a given point P , $\Omega(n \log n)$ steps are required by any algebraic decision tree that can decide if there is an equitable two-fan for S with center at P .*

Given n points in general position in R^2 , Willard [24] asked for a pair of non-parallel lines ℓ_1 and ℓ_2 that equitably partition the points; i.e., in each of the four open quadrants they define, there are at most $n/4$ points. An efficient algorithm for this was implied by results in Cole, Sharir, and Yap [14], and an optimal $O(n)$ algorithm follows immediately using Megiddo's separated, discrete ham-sandwich cut [22]. In fact we can even insist that the lines are orthogonal: this is implied by another result of Bárány and Matoušek [8] that again uses Borsuk-Ulam along with equivariant topology. Here we give an easy, direct combinatorial proof that there is an orthogonal four-partitioning, and we describe the computational complexity of finding one. Specifically we prove

Theorem 1 *Given a set S of n points in general position in R^2 , there exist orthogonal lines ℓ_1 and ℓ_2 that equipartition S , and they may be found in $\Theta(n \log n)$ RAM steps.*

On the other hand, in the special case where the n points of S are in convex position, their radial order is the same from every point interior to $\text{conv}(S)$. This fact simplifies the problem enough to allow us to find an orthogonal equipartitioning in linear time.

Lemma 2 *An equitable orthogonal partitioning for n points in convex position can be found in time $O(n)$.*

We discuss several open problems, particularly algorithmic issues for (i) equipartitioning three sets with a two fan, (ii) equipartitioning two sets with a three fan, (iii) balanced partitioning of a set in R^3 by three planes, and (iv) splitting a necklace with three types of jewels between two thieves; in addition we mention some outstanding combinatorial problems.

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