Efficient Algorithm for Approximating Maximum Inscribed Sphere in High Dimensional Polytope

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Abstract

In this paper, we consider the problem of computing a maximum inscribed sphere inside a high dimensional polytope formed by a set of halfspaces (or linear constraints), and present an efficient algorithm for computing a $(1-\epsilon)$-approximation of the sphere. More specifically, given any bounded polytope P defined by n d-dimensional halfspaces, an interior point O of P, and a constant $\epsilon > 0$, our algorithm computes in $O(nd/\epsilon^3)$ time a sphere inside P with a radius no less than $(1-\epsilon)R_{opt}$, where R_{opt} is the radius of a maximum inscribed sphere of P. Our algorithm is based on the core-set concept and a number of interesting geometric observations. Our result settles an open problem posted by Khachiyan and Todd [4] for the case of spheres.

1 Overview

In this paper, we consider the following problem of approximating maximum inscribed sphere (MaxIS): Given a set $\mathcal{H} = \{H_1, H_2, \cdots, H_n\}$ of halfspace in d-dimensional Euclidean space E^d, a point O in the common intersection P of $\mathcal{H}$, and a small constant $\epsilon > 0$, compute a sphere B inside P (assume that P is bounded) with a radius no less than $(1-\epsilon)R_{opt}$, where R_{opt} is the radius of a maximum inscribed sphere of P.

Efficiently computing the maximum inscribed sphere or ellipsoid in high dimensional polytope formed by a set of halfspaces (or linear constraints) is a challenging problem in theoretical computer science and operations research. It is closely related to the interior-point method for linear programming (LP). One way [6] to solve the MaxIS problem is to reduce it to a linear programming (LP) problem with one more dimension and then uses existing LP algorithms to solve it. Thus optimally solving the MaxIS problem could be rather costly (i.e., with the same complexity as an LP problem). With the power of core-sets [1], it has been shown that many problems (such as the clustering problems and shape fitting problems [1]) in high dimensional space which originally have high time complexities can now be efficiently solved if only approximations are sought. For instance, using core-sets, a $(1+\epsilon)$-approximation of the minimum enclosing sphere (MinES) of a set of n points in d-dimensional space can be computed in linear time [1]. Thus it would be very interesting to know whether the core-set concept can be used to speed up the computation of the MaxIS problem.

A closely related problem of the MaxIS is the problem of computing maximum inscribed ellipsoid (MaxIE). Due to its direct applications in the interior-point method, the MaxIE has received a great deal of attention in the past. Khachiyan and Todd [4] built a polynomial bound for computing an approximate point of the center of the maximal inscribed ellipsoid in the polytope defined by linear constraints. Most recently, Anstreicher [2] showed that computing a $(1-\epsilon)$-approximation of the maximum inscribed ellipsoid of the polytope can be done in $O(n^{3.5}\log(\frac{nR}{\epsilon}))$ time, where R is the known aspect ratio of the polytope. This is the best complexity to our knowledge. For the case $n^2 \gg d$, Zhang and Gao [7] recently obtained an improvement over Khachiyan and Todd's algorithm.

A dual problem of MaxIE is that of computing a minimal enclosing ellipsoid (MinEE) for the convex hull of n points in E^d. Khachiyan [3] showed that a $(1+\epsilon)$-approximation of the MinEE can be computed in $O(n^{3.5}\log(\frac{n}{\epsilon}))$ time. Recently, Kumar and Yildirim designed an approximation algorithm by using core-sets [5]. They proved the existence of a core-set of size at most $\alpha = O(d(\log d + \frac{1}{\epsilon}))$. The complexity of their algorithm is $O(nd^2\alpha + \alpha^{4.5}\log(\frac{\alpha}{\epsilon}))$, which is linear in terms of n.

In this paper we present an efficient algorithm for computing a $(1-\epsilon)$-approximation of the MaxIS of P. The running time of our algorithm is $O(\frac{dn}{\epsilon^3})$, which is linear in term of the size of the input (i.e. nd), and does not depends on the aspect ratio of the polytope. Our algorithm first translates the origin of the coordinate system to the interior point O, and then use dual transform to map each halfspace H_i to a point H_i^* in the dual space. By superimposing the dual space onto the primal space so that the two spaces share the same coordinate system, we can relate the MinES of the dual points to the MaxIS of the halfspaces in the same space. We show that when the center C of the MinES overlaps the origin O, the maximal sphere centered O is the maximum inscribed sphere. Thus, to solve the MaxIS problem, we only need to make the center C close enough to the origin O. Our main idea for making C approach O is to move O in certain direction by a distance s while keeping all halfspaces at their original positions. In this way, all the dual points will change accordingly and more importantly the center C of the new dual points will move close to O. Using this movement of O as a basic step, we can iteratively improve the maximal inscribed sphere centered at O.

Our algorithm distinguishes two cases: (a) The starting point O is close to the optimal point; (b) O is at arbitrary position in the polytope of $\mathcal{H}$. For case (a), we are able to show that by selecting a good value for s, each basic step reduces the radius of the MinES significantly and in total it takes only $O(1/\epsilon^2)$ steps to obtain a $(1-\epsilon)$-approximation of the MaxIS. For case (b), we first construct a $(d+1)$-dimensional polytope P_{d+1} from the polytope P of $\mathcal{H}$, and then map the starting point O of P to a good starting point in P_{d+1}. After obtaining a good approximation of the MaxIS in P_{d+1}, we can map the center back to P. We are able to show that the point in P is now a good starting point for P.

Our algorithm can be easily implemented and converges very quick in practice. It takes only a small constant number of steps for the MinES to converge to its $(1-\epsilon)$-approximation even for dimensions as high as 1000. Our algorithm settles the open problem of Khachiyan and Todd [4] for the case of spheres.

References

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