The Complexity of Diffuse Reflections in a Simple Polygon

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Abstract. The complexity of the visibility region formed by a point light source after k diffuse reflections in a simple n-sided polygon is $O(n^9)$, which is the first result polynomial in n, with no dependence on k. This bound is an exponential improvement over the previous bound of $O(n^{2\lceil (k+1)/2 \rceil + 1})$ due to Prasad et al. [8].

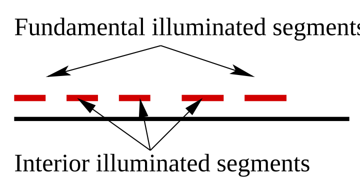
Introduction. Visibility problems in computational and combinatorial geometry have been studied extensively (see [3, 6, 9] and references therein). We confine our attention to results in the plane, more specifically those referring to visibility inside a simple polygon P with n vertices. Two points are *visible* to each other if the segment connecting them is contained in the polygon. The region visible from a point in P is a star-shaped polygon with at most n edges. The set of points of P visible from at least one point of a segment in P (the so-called "weak visibility polygon" from a segment) is a simple polygon with $O(n)$ edges and can be computed in linear time [5].

Aronov et al. [2, 1] and Davis [4] initiated the study of complexity of the region lit up by a single source of light in a simple polygon if reflection is allowed. Two models are considered. In both of them, any light incident upon a polygon corner is absorbed rather than reflected. In the *specular* reflection model, a light ray incident on a point in the interior of a polygon edge is reflected, as in geometric optics, with the angle of reflection equaling the angle of incidence. In the *diffuse* model which we consider in this paper, the light ray incident upon an interior point of an edge reflects in all possible interior directions. Aronov et al. [2] argue that for both diffuse and specular reflection the maximum complexity of the region lit up by a point light source with one reflection allowed is $\Theta(n^2)$. The results were generalized in [1] to any number k of reflections and it was shown that for specular visibility this complexity is $O(n^{2k})$ and that this bound is tight for constant k. The case of multiple diffuse reflection is discussed by Prasad et al. [8], where they gave a bound of $O(n^{2\lceil (k+1)/2 \rceil + 1})$ on the complexity of the region lit up by a point with at most k diffuse reflections. Surprisingly, even though this bound is exponential in k (for arbitrarily large n), no constructions were known for diffuse reflection with complexity $\omega(n^2)$, irrespective of the number of reflections used. This gave rise to the conjecture in [8] that this in fact is the correct answer, for $k \geq 1$ reflections. As the analysis in [2], among other things, proves that the region visible from a point with one diffuse reflection is always simply connected, it has been suggested that this remains true when more diffuse reflections are allowed. However, Pal [7] gives an example when this conjecture fails already when two reflections are allowed.

In this paper, we partially settle the former conjecture on multiple diffuse reflections, namely we argue that the complexity of the region visible from a point with at most k diffuse reflections is $O(n^9)$, for any value of k.

Main Result. We sketch the proof of the main result. For details, see the full version.

One approach to bound the total complexity of the illuminated region at time k is to estimate the total complexity of the illuminated segments at time $k-1$. We prove that if x segments are illuminated in a polygon with n edges at time $k-1$, then the complexity of the illuminated region of the polygon (including the interior) at time k is $O(nx^2)$. In this paper, we first find the total complexity of the illuminated segments at all time and then use the result to find the total complexity of the illuminated region at time k. Because the entire polygon is illuminated at time $k = n$, we only consider $k < n$, since beyond this time there is no additional complexity.



We classify illuminated segments into two categories — fundamental segments and interior segments. A fundamental segment is defined to be the first segment from either endpoint of an edge. All non fundamental segments are considered interior segments, each lying between two illuminated segments. All segments are disjoint. The complexity of the illuminated segments at time k is the number of fundamental and interior segments at time k. The complexity of the illuminated segments at all times is the sum of the number of fundamental and interior segments at all times.

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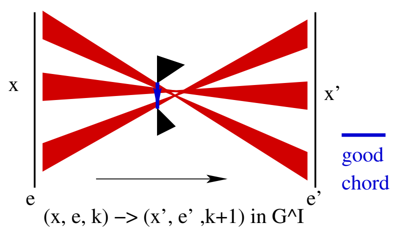
Bounding the number of fundamental segments is trival because there are at most 2 fundamental segments on each edge at one time. The total number of fundamental segments on all edges at all times is $2n^2$.

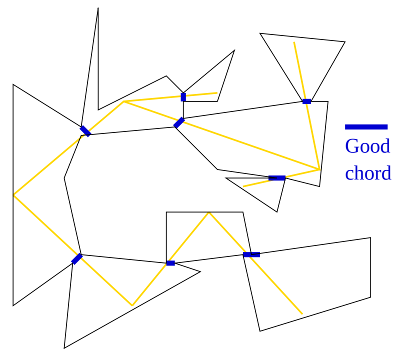
Bounding the number of interior segments at all times is the core of this paper. We reduce the problem to counting the number of nodes in a directed graph, G^I. Each node in G^I is an interior triple, (x, e, k), which is intrepreted as "the interior segment x is illuminated on edge e at time k." There is an edge from (x, e, k) to $(x', e', k+1)$ iff 1) the interior segment x on edge e illuminates the interior segment x' on edge e' at time $k+1$ and 2) the two adjacent segments of x also illuminate two segments on e' at time $k+1$. See the figure below:



Therefore, a source node (y, e', k) in G^I (a node with no incoming edges) is illuminated by either 1) a fundamental segment or 2) an interior segment, say x, but not both adjacent segments of x illuminate a segment on the edge e'. Finding the total number of interior segments at all time is equivalent to finding the number of nodes in G^I. We need to find the number of source node in G^I and the number of nodes reachable from each source node in G^I.

In the paper, we prove that there are at most $4n^3$ source nodes in G^I and that at most $n-1$ nodes in G^I are reachable from each of those nodes. The general idea for proving the second fact is as follows: There is one big observation: each edge in G^I must go through some chords. Exactly one of those chords is considered as a good chord. We prove that if a light of an interior triple passes through a good chord, it will never go through the same good chord in the opposite direction again. Also, once the light splits through two different good chords, it cannot further interact. See the figure below.

Therefore, no two nodes reachable from a source node have the same e value. Therefore, at most $n-1$ nodes can be reachable from each source node. Since there are at most $4n^3$ source nodes, the total number of nodes in G^I (the total number of interior segments) is at most $4n^4$. The total complexity of the illuminated segments over all time is the sum of total number of fundamental and interior segments, which is $O(n^4)$. Finally, we prove that if the complexity of the illuminated edges over all time is $O(n^4)$, the total complexity of the illuminated region at time k is $O(n^9)$



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