Bounding the Number of Plane Graphs

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We investigate the number of plane geometric, i.e., straight-line, graphs, a set S of n points in the plane admits. We show that the number of plane graphs is minimized when S is in convex position, and that the same result holds for several relevant subfamilies.

In addition we construct a new extremal configuration, the so-called double zig-zag chain. Most noteworthy this example bears $\Theta^*(\sqrt{72}^n) = \Theta^*(8.4853^n)$ triangulations and $\Theta^*(41.1889^n)$ plane graphs (omitting polynomial factors in both cases), improving the previously known best maximizing examples.

1 Introduction

Let us denote by $\mathcal{S}_n$ the set of sets of n points in the plane in general position, that is, no three points of a set $S \in \mathcal{S}_n$ lie on a common line. With $\Gamma_n \in \mathcal{S}_n$ we denote any set of n points in convex position. Throughout this paper we consider *plane geometric graphs* G on top of $S \in \mathcal{S}_n$. That means that the set of vertices of G is S, edges of G are straight-line segments spanned by vertices of S and two edges of G do not intersect in their interior but might have endpoints in common.

In other words, we consider the rectilinear drawing of the complete graph K_n with vertex set $S \in \mathcal{S}_n$ and study its crossing-free subgraphs. The problem of how large the number of such subgraphs may be has been attracting a lot of attention; many references can be found in [15] and in the lately published book [10]. It has also been proved recently that the set of crossing-free subgraphs can be realized as a polytope [17].

A fundamental contribution was given by Ajtaj et al. [9]: the number of plane graphs on top of any $S \in \mathcal{S}_n$ is bounded from above by some fixed exponential c^n; the bound $c \leq 10^{13}$ was given there and has been successively improved up to $c \leq 472$. It is worth mentioning that a main tool developed in [9] is the nowadays famous "Crossing Lemma": every planar drawing of a graph with n vertices and $m > 4n$ edges contains at least cm^3/n^2 crossings, for some constant c. This result, independently proved by Leighton [16], has later found many applications. In fact, the motivation in [9] was to provide an upper bound *for the number of polygonizations* (crossing-free spanning cycles) on top of any $S \in \mathcal{S}_n$. Obviously the bound for generic plane graphs applies, yet better specific bounds have been obtained for polygonizations as well as for plane triangulations, perfect matchings, spanning trees and many other classes of plane graphs; precise references are given later in this paper.

To describe the asymptotic growth of the number of graphs we use the $\mathcal{O}^*()$-, $\Omega^*()$-, and $\Theta^*()$-notation. In these notations we neglect polynomial factors and just give the dominating exponential term. Moreover when the base of the exponent is explicitly given as a numerical value, this has to be seen as an approximation up to the given precision.

Maximal plane graphs, i.e., triangulations, are a case of special interest, because any plane graph can be completed to a triangulation and hence any upper bound $\mathcal{O}^*(\alpha^n)$ on the number of triangulations implies a corresponding upper bound $\mathcal{O}^*(2^{3n}\alpha^n) = \mathcal{O}^*((8\alpha)^n)$ on the number of generic plane graphs, because every triangulation has at most $3n-6$ edges and therefore contains at most 2^{3n} subgraphs. The current best upper bound for triangulations is $\mathcal{O}^*(59^n)$ and was obtained by Santos and Seidel in [20]; the aforementioned bound of $\mathcal{O}^*(472^n)$ for plane graphs is derived from that.

On the opposite direction, it is also known that every $S \in \mathcal{S}_n$ admits at least $\Omega^*(2.33^n)$ triangulations, and it has been conjectured that the number of triangulations is minimized when S is the point set called *double circle*, that has $\Theta^*(\sqrt{12}^n)$ triangulations [5].

In this work we obtain new lower and upper bounds for the maximum and minimum, respectively, number of plane geometric graphs of different types. All given bounds are exponential bounds of the form α^n where the goal is to optimize the base α.

More precisely, we prove that the number of plane graphs of several classes (including the whole one) is minimized by point sets in convex position, a fact that was known for perfect matchings, spanning trees and spanning paths [14, 21]. Here we provide a unified ap-

Type	Lower Bound	Number for Γ_{10}	Upper Bound	
spanning cycles: $\mathsf{sc}(n)$	1	1	$\Omega^*(4.64^n)$ [14]	$\mathcal{O}^*(94^n)$ [21]
perfect matchings: $\mathsf{pm}(n)$	$\Theta^*(2^n)$ [14]	42	$\Omega^*(3^n)$ [14]	$\mathcal{O}^*(10.04^n)$ [21]
spanning paths: $\mathsf{sp}(n)$	$\Theta^*(2^n)$	1 280	$\Omega^*(4.64^n)$ [14]	$\mathcal{O}^*(100.81^n)$ [21]
triangulations: $\mathsf{tr}(n)$	$\Omega^*(2.33^n)$ $\mathcal{O}^*(3.47^n)$ [5]	250	$\Omega^*(8.48^n)$ [*]	$\mathcal{O}^*(59^n)$ [20]
pointed pseudo-tri.: $\mathsf{ppt}(n)$	$\Theta^*(4^n)$ [4]	1 430	$\Omega^*(12^n)$ [8]	$\mathcal{O}^*(177^n)$ [19]
pseudo-triang.: $\mathsf{pt}(n)$	$\Theta^*(4^n)$ [4]	1 430	$\Omega^*(20^n)$ [8]	$\mathcal{O}^*(177^n)$ [19]
spanning trees: $\mathsf{st}(n)$	$\Theta^*(6.75^n)$ [13]	246 675	$\Omega^*(10.42^n)$ [11]	$\mathcal{O}^*(314.7^n)$ [18]
cycle-free graphs: $\mathsf{cf}(n)$	$\Theta^*(8.22^n)$ [13]	2 117 283	$\Omega^*(11.09^n)$ [14]	$\mathcal{O}^*(398.25^n)$ [*]
connected graphs: $\mathsf{cg}(n)$	$\Theta^*(10.39^n)$ [13]	5 616 182	$\Omega^*(35.49^n)$ [*]	$\mathcal{O}^*(472^n)$ [14]
all plane graphs: $\mathsf{pg}(n)$	$\Theta^*(11.65^n)$ [13]	21 292 032	$\Omega^*(41.18^n)$ [*]	$\mathcal{O}^*(472^n)$ [14]

proach that encompasses those results and extends to many more classes. We also study some upper bounds and, in particular, we prove the existence of a certain point set that has $\Theta^*(\sqrt{72}^n) = \Theta^*(8.4853^n)$ triangulations and $\Theta^*(41.1889^n)$ plane graphs, improving the previously known best maximizing examples and disproving the common belief that the tight upper bound for the number of triangulations would be $\Theta^*(8^n)$.

Several results are shown in the enclosed table, where the reference [*] stands for this work.

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