Fold-and-Cut
Project Survey Report

Parasaran Raman
praman@cs.utah.edu

7-May-2009
Abstract

Given a polygonal flat origami on a paper, can we make a series of folds such that after making a single straight line cut on the folded structure, we can retrieve back the given polygonal shape? The fold-and-cut problem was formally stated by Martin Gardner in 1960. The problem has a much longer history, going back to 1721 in a Japanese puzzle book, Betsy Ross in 1777, and Houdini in 1922. The fold-and-cut problem is analogous to the problem of, given a planar graph drawn with straight edges on a piece of paper, can the paper be folded at so as to map the entire graph to a common line, and map nothing else to that line? It is quite surprising that this is always possible, for any collection of line segments in the plane, forming non-convex polygons, adjoining polygons, nested polygons, etc. There are two solutions to this problem. The first solution is based on a structure called the straight skeleton, which captures the symmetries of the graph, thereby exploiting a more global structure of the problem. The second solution is based on disk packing to make the problem more local, and achieves and bound on the number of creases.
1. History

Origami Mathematics is the study of geometry and the properties of paper folding. The first published reference to folding and cutting is a Japanese book, *Wakoku Chiayekurabe*, by Kan Chu Sen, published in 1721. This book contains a variety of very interesting problems. There is one particular problem that requires us to fold a rectangular piece of paper flat and make one complete straight cut, so as to make a typical Japanese crest called *sangaibisi*. The author gives a solution consisting of a sequence of simple folds, each of which folds along a line.

Another reference is that of *Betsy Ross and her five-pointed star*. In 1777, George Washington and a committee of the Congress showed Betsy Ross plans for a flag with thirteen six-pointed stars, and asked whether she could make such a flag. She said that she would be willing to try, but suggested that the stars should have five points.

To support her argument, she showed how easily such a star could be made, by folding a sheet of paper and making one cut with scissors. The committee decided to accept her changes, and George Washington made a new drawing, which Betsy Ross followed to make the first American flag.

Folding and cutting may have been used for a magic trick by Houdini, before he became a famous escape artist. His 1922 book *Paper Magic* (E. P. Dutton & Company, pages 176-177) describes a method for making a five-pointed star.

Another magician, Gerald Loe, studied the fold-and-cut idea in some detail; his 1955 book *Paper Capers* (Magic, Inc., Chicago) describes how to cut out arrangements of various geometric objects, such as a circular chain of stars. Martin Gardner wrote about the fold-and-cut problem in his famous series in *Scientific American* (“Paper cutting”, chapter 5 of *New Mathematical Diversions (Revised Edition)*, Mathematical Association of America, Washington, D.C., 1995.)
2. One complete straight line cut

Any planar straight line drawing (collection of straight edges) can be cut out of a sufficiently large paper by a single straight line cut after appropriate folding. It is surprising that the polygonal shapes need not be connected. It can include adjoining polygons, nested polygons, floating line segments and points.

The algorithm by Demaine, Demaine and Lubicw computes a crease pattern that produces a flat origami. To understand the scheme better, it becomes necessary to defines the following terminologies.

- **Plane Graph**
  A planar graph with every edge straight and of positive length is a plane graph. Every pair of edge intersects at only one vertex.

- **Crease Pattern**
  A simple plane graph is a crease pattern.

- **Folding / Origami**
  Folding with the crease pattern is a continuous function from $\mathbb{R}^2$ to $\mathbb{R}^3$ with no crossings. This function maps every face to a congruent copy in three dimensions.

- **Flat Origami**
  The origami whose image lies on a plane is a flat origami.

- **Mountain-Valley Assignments**
  These are specific assignments to edges in the crease pattern. A folding angle of $\Pi$ ($180^\circ$) is called a Mountain and a folding angle of -$\Pi$ (-180$^\circ$) is called a Valley. This is relative to the top side of the paper where the polygonal shape is laid out.
• **Cut Graph**

The plane graph that is provided is called the cut graph and the corresponding vertices, edges and faces are called cut vertices, cut edges and cut faces respectively.

The following theorem gives us a better understanding of the whole problem.

<table>
<thead>
<tr>
<th>Theorem 1:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Given a cut graph and a side assignment, then there exists a flat folding of the plane that maps the cut edges to a common line and appropriately maps faces above or below this line.</td>
</tr>
</tbody>
</table>

According to this theorem, we would like a 2-coloring of the cut graph where the colors represent above and below.

We shall now describe the technique for creating a basic crease pattern and the assignment of mountains and valleys to the folds.

**3. Straight Skeleton**

The intuition behind the straight skeleton method is this: to line up two edges, a fold is made along the bisector of their extensions. The straight skeleton consists of several line segments, each of which is a subsegment of a bisector of two graph edges.

The straight skeleton consists of the majority of the creases and achieve the desired lining up of cuts. In addition, there are perpendicular creases. From each vertex of the straight skeleton, we shoot a ray perpendicular to each reachable cut edge, and the ray bounces (reflects) through any skeleton edges it meets.

The straight skeleton is defined by a continuous shrinking process in which the edges of the polygon are moved inwards parallel to themselves at a constant speed. As the edges move in this way, the vertices where pairs of edges meet also move, at speeds that depend on the angle of the vertex.
If one of these moving vertices collides with a nonadjacent edge, the polygon is split in two by the collision, and the process continues in each part. The straight skeleton is the set of curves traced out by the moving vertices in this process.

Let $P$ be the given polygonal shape. Imagine that the boundary of $P$ is contracted towards $P$'s interior, in a self-parallel manner and at the same speed for all edges. Lengths of edges might decrease or increase in this process. Each vertex of $P$ moves along the angular bisector of its incident edges. This situation continues as long as the boundary does not change topologically. There are two possible types of changes:

- **Edge event**
  An edge shrinks to zero, making its neighboring edges adjacent now.

- **Split event**
  An edge is split, i.e., a reflex vertex runs into this edge, thus splitting the whole polygon. New adjacencies occur between the split edge and each of the two edges incident to the reflex vertex.

After either type of event, we are left with a new, or two new, polygons which are shrunk recursively if they have non-zero area. The straight skeleton, $S(P)$, is defined as the union of the pieces of angular bisectors traced out by polygon vertices during the shrinking process. $S(P)$ is a unique structure defining a polygonal partition of $P$. Each edge $e$ of $P$ sweeps out a certain area which we call the face of $e$. Bisector pieces are called arcs, and their endpoints which are not vertices of $P$ are called nodes, of $S(P)$.

The plane polygonal shape is represented as a cut graph. Now, the problem is to find the flat folding of the paper and a cut line, such that the intersection of the folding with the cut line is exactly the cut graph.

The following lemmas and definitions are useful in understanding the straight skeleton.
Lemma 1
The straight skeleton has $O(n)$ vertices, edges and faces.

Lemma 2
Every Skeleton contains exactly one cut edge. Every cut edge is contained in exactly one cut edge.

Lemma 3
If $e$ is a skeleton edge, $f_1$ and $f_2$ are incident skeleton faces, and $c_1$ and $c_2$ are the cut edges in $f_1$ and $f_2$, then $e$ bisects $c_1$ and $c_2$.

We determine convex and reflex portions of the skeleton edges to identify the mountain-valley assignments to the folds. The mountain-valley assignments basically determine to which side the paper should be folded. Let $c'_1$ and $c'_2$ be the extensions of the cut edges and $e'$ the extension of the skeleton edge.

- If $c'_1$ and $c'_2$ are distinct and parallel, and $e'$ lies between them, then all of $e$ is considered to be convex.
- If $c'_1$ and $c'_2$ are the same line, and $e'$ is perpendicular to them, then all of $e$ is considered to be reflex.
- If $e'$ bisects a certain wedge $W(e, c'_1, c'_2)$, the portion of $e$ inside the closed wedge is convex and the portion in the closed complement is considered to be reflex.

The cut edges form the first few folds. We will have a valley fold if it occurs between two faces above the cut line. A mountain fold is formed when if the two faces are below the cut line. In order to form the folds on the straight skeleton, we have to add the folds perpendicular to each cut edge $e$. Let us now look at various components that form in the process of constructing crease patterns.
3.1. Perpendiculars

The straight line is not foldable by itself since the vertices can have degree three. A fold perpendicular to the cut edge solves this and maintains the property of the cut edge line up.

*The perpendicular associated with any point consists of line segments, rays and lines (perpendicular edges) – each associated with a skeleton face.*

There are two types of perpendiculars.

• **Real Perpendicular**
  
  The perpendiculars that are incident to a skeleton vertex are real perpendiculars.

• **Imaginary Perpendicular**
  
  All other perpendiculars belong to this category.

  Lets now relate to a lemma that defines a bound on the number of real perpendiculars.

<table>
<thead>
<tr>
<th><strong>Lemma 4</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>There are $O(n)$ real perpendiculars.</td>
</tr>
</tbody>
</table>

An interesting observation is that the perpendicular edges associated with skeleton face $f$ is perpendicular to the cut edge contained in $f$.

3.2. Spirals

One interesting phenomenon with perpendiculars is that they can cause spiraling. The cut graph can be an infinite pinwheel as shown in the figure below.
However we have the below lemma that bounds the number of real perpendicular edges.

**Lemma 5**

*Any bounded region of the plane is intersected only by finite number of real perpendicular edges.*

![Cut graph](image)

**Figure 1.** The cut graph (thick lines) is an infinite pinwheel. Each perpendicular (dashes) consists of infinite number of edges

### 3.3. Corridors

All of the perpendicular edges form the perpendicular graph. Corridors are the faces of these perpendicular graphs. These are regions composed only by perpendiculars. There are two kinds of corridors, namely,

- **Linear**
  The interior of the linear corridor is homeomorphic to an infinite band.

- **Circular**
  The interior of the circular corridor is homeomorphic to an annulus.
Lemma 6

Every corridor C has a constant width and is either linear or circular and has wither one or two walls.

A wall is one of the perpendiculars that bound the corridor.

4. Folding along the creases

We now have to start the process of folding along the creases that we have made so far. The problem is split into two parts,

• Fold a single corridor into an accordion using alternate mountain and valley assignment. When a corridor C folds into an accordion, the cut edges intersecting C line up and each wall of C lines up.

• In step 2, we try to join all the corridors through a Tree Model.

Accordion is a folded state. Each corridor folds by itself into this simple structure called accordion. Creases in a subdivided corridor alternate between mountain and valley.

Lemma 7

Two adjacent corridors fold into accordions that match up at their common side.

In this way, the accordions are folded locally. They can be joined by lining them up at the cut edges.

Each accordion lies in a vertical strip resulting in a 1-D structure called shadow tree. Any flat folding of the shadow tree corresponds to a flat folding of the crease pattern that uses all the creases except some of the perpendicular edges. Each of these trees has a flat folding. So, for any plane graph inducing linear corridors, there is a flat folding that uses all creases except some of the perpendicular edges.
The problem now is to fold at these joins so that accordions line on a common plane. This produces the desired flat origami. This can be modeled as a tree in 2-D. The tree corresponds to xy projection of the folded model. The accordion is represented by the edges and the sides of each accordion is represented by the vertices.

**Theorem 2**

*Given a cut graph and any side assignment, there exists a flat folding of the plane that maps the cut edges to a common line and appropriately maps the faces above or below this line.*

![Theorem 2 Diagram](image)

**Figure 2.** Left – Full crease pattern of a spiral polygon; Center – Shaded corridor is folded into an accordion; Right – The tree model for the folding

5. **Treemaker 5**

Treemaker 5 is a computer program that designs a non-trivial origami figure based on a description of the number, lengths, and connectedness of the flaps written by Robert J. Lang. We would need to draw a stick figure of the base on the screen; each stick in the stick figure will be represented by a flap on the base. Once you have defined the tree, TreeMaker computes the full crease pattern for a base which, when folded, will have a projection equivalent to that specified by the defining tree.

The version 4 of treemaker did not have the procedure to compute the crease assignment (mountain or valley), but with a few simple rules and some exploration by
hand, the proper crease assignment can usually easily be found.

- All ridge creases are valley creases;
- All gusset creases are mountain creases;
- Most axial creases are mountain creases.

But Treemaker 5 solves this problem as it computes the full mountain-valley assignments as well. Treemaker 5 source code is available online and I have started to play around with it to create various folds and cuts.

6. Conclusion

We can create crease patterns and thus a flat origami that lines up a given plane graph. This allows one to fold a sheet of paper flat and make one straight line cut to create any desired pattern of cuts. Folding and Cutting is thus sufficient to create any plane graph.

7. References

- Erik D. Demaine, Martin L. Demaine, Anna Lubiw, Folding and cutting paper. JCDCG '98
- Oswin Aichholzer, Franz Aurenhammer, David Alberts, and Bernd Gärtner, "A Novel Type of Skeleton for Polygons", JUCS
- Geometric Folding Algorithms, Erik Demaine & Joseph O'Rourke, Cambridge Press
- R.J. Lang, Treemaker 5.0: A Program for Origami Design, 2004
- The images used in the report are borrowed from Eric Demaine's JCDCG '98 paper.