Tetrahedron Meshes

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Overview

- Fundamental Tetrahedron meshing
  - Meshing polyhedra
  - Tetrahedral shape
  - Delaunay refinements
  - Removing silver

- Variational Tetrahedral Meshing
Meshing Polyhedra

- Is that easy?
- A *Polyhedron* is union of convex polyhedra

\[ P = \bigcap_{i \in I} H_i \]
Fig. 47. A non-convex polyhedron

- Faces are not necessarily simply connected
Facets and Segments

- Neighborhood of $x$
  \[ N_\varepsilon = \left( x + \varepsilon \cdot b \right) \cap P \]
  - $b$: open unit ball

- Face figure of $x$
  \[ x + \square_{\lambda > 0} \lambda \left( N_\varepsilon (x) - x \right) \]

- Face of $P$: closure of maximal collection of point with identical face figures
What does it look like?

1- Faces
   - Face figures a line
   - Polyhedra segment

2- Faces
   - Face figure a plane
   - Polyhedra facets
- 24 vertices, 30 segments, 11 faces
  - 6 segments and 3 faces are not non-connected
Tetrahedrization

- Tetrahedrization of P is a simplicial complex K whose underlying space is P: $|K|=P$
  - Only bounded polyhedra have tetrahedrization
  - Every vertex of P is vertex of K

- Is the number of vertices of K = P?
Is that easy?

- Prim – easy: 3 tetrahedrons
- The Schonhardt polyhedron? How many?
Ruppert and Seidel: the problem of deciding if a polyhedron can be tetrahedrized without inserting extra vertices is NP-hard and problem of deciding if a polyhedron can be tetrahedrized with only $k$ additional vertices is NP-hard.
Every bounded polyhedron has a tetrahedration
Fencing algorithms

- Step1: Erect the fence of each segment.
- Step2: Triangulate the bottom facet of every cylinder and erect fences from the new segments.
- Step3: Decompose each wall into triangles and finally tetrahedrize each cylinder by constructing cones from an interior point to the boundary.
Limits

- Upper bound: $28m^2$ - number of tetrahedrons for $m$-segments Polyhedra
- Lower bound: $(n+1)^2$ with $n$ : number of cut $m=14n+8$

Fig. 52. Polyhedron $Q$ with two families of cuts almost meeting along the saddle surface
Tetrahedral Shape

- What is good and bad tetrahedra?
- Bad tetrahedra

Wedge

Needle

Cap

Sliver
Bad tetrahedral classification

- **Skinny**: vertices are close to a line

- **Flat**: vertices are close to a plane

Fig. 53. Five fuzzy types of skinny tetrahedra

Fig. 54. Four fuzzy types of flat tetrahedra
Quality Metrics

- **2D**: minimum angle in the triangulation
- **3D**:
  - **Radius-edge ratio**
    \[ \rho = \rho(\sigma) = \frac{R}{L} \]
    - R: outer radius
    - L: shortest edge
  - **Aspect ratio**
    \[ \vartheta = \vartheta(\sigma) = \frac{R}{r} \]
    - r: inner radius
  - **Volume ratio**
    \[ \varphi(\sigma) = \frac{V}{L^3} \]
    - V: volume
Ratio properties

- A mesh of tetrahedra has *ratio property* if \( \rho \leq \rho_0 \) for all tetrahedra
  - For all triangles \( \rho \leq \rho_0 \)

- Characteristics:
  - *Claim A:* if abc is a triangle in K then
    \[
    \frac{1}{2\rho_0} \|a - b\| \leq \|a - c\| \leq 2\rho_0 \|a - b\|
    \]
    - Proof: geometry
      - Ratio properties
        \[
        \frac{R}{\rho_0} \leq \|a - c\| \leq 2R
        \]
        \[
        2R \geq \|a - b\| \geq \frac{R}{\rho_0}
        \]
Ratio property

- Denote $\eta_0 = \arctan 2\left(\rho_0 - \sqrt{\rho_0^2 - 1/4}\right)$

- Claim B: if angle between $ab$ and $ap$ less than $\eta_0$ then
  $$\frac{1}{2} \|a - b\| \leq \|a - p\| \leq 2\|a - b\|$$

Proof:

$$\|a - v\| = R - \sqrt{R - \|a - b\|^2 / 4}$$

$$= \frac{\|a - b\|^2 / 4}{R - \sqrt{R - \|a - b\|^2 / 4}} \geq \left(\rho_0 - \sqrt{\rho_0^2 - 1/4}\right)\|a - b\|$$

$$\angle bax = \arctan 2\left(\frac{\|x - v\|}{\|a - b\|}\right)$$

$$\geq \arctan 2\left(\rho_0 - \sqrt{\rho_0^2 - 1/4}\right)$$

$$= \eta_0$$
Length Variation and Constant degree

- **Length variation**: if $ab$, $ap$ are edges in $K$

  \[ \frac{||a - b||}{v_0} \leq ||a - p|| \leq v_0 ||a - b|| \]

  - Lengths of edges with common endpoint is bound

- **Constant degree**: Every vertex $a$ in $K$ belong to at most

  \[ \delta_0 = \left(2v_0^2 + 1\right)^3 \]

  - However this constraint is too loose
Delaunay Triangulation

Canonical, associated to any point set.
Delaunay Triangulation

- Dual to Voronoi Diagram
- Connect Vertices Across Common Line
Properties of Delaunay triangulation

- maximizing the minimum angle in the triangulation:
  - Better lighting performance
  - Improvements on the accuracy
- Even stronger: lexicographically maximized sequence of angles
- Every triangle of a Delaunay triangulation has an empty circumcircle
Every tetrahedra of a Delaunay Tetrahedration has an empty circumsphere. However:
- max-min angle optimality in 2D doesn’t maintain
- Tetrahedra can have small dihedral angles.
Left: Delaunay tetrahedralization: have arbitrarily thin tetrahedron known as a *sliver*

Right: non-Delaunay tetrahedralization
Edge flip

Incremental algorithms in 2D: edge flips

In 3D: it is not so easy
Circumcircle as Orientation

- The Circumcircle test is an orientation test
- Let $p' = (p, \|p\|^2)$
Properties of the Space of Spheres

- Points below the cutting plane are inside the circle
- Also applies to triangular regions
Using Convex Hulls

Simple Algorithm

1. Project onto paraboloid.
2. Compute convex hull.
3. Project hull faces back to plane.
Advantages of Convex Hull Approach

- Many good Convex Hull Algorithms (e.g. QuickHull)
- Simple extension to arbitrary dimensions
- No strange infinite triangle initialization
Problems

- Segment that not cover by the edges
- Faces are not covered by triangles of D
  Have to add new points and update Delaunary tetrahedration
- well-spaced points generate only round or sliver Delaunay tetrahedra
Delaunary refinement

- Construct Delaunary tetrahedration has ratio property

- Definition
  - encroached upon sub – segment
  - encroached upon sub – facet
Refinement algorithms

- **Rule1**: If a subsegment is encroached upon, we split it by adding the mid-point as a new vertex to the Delaunay tetrahedrization.

- **Rule2**: If a subfacet is encroached upon, we split it by adding the circum-centre $x$ as a new vertex to the Delaunay tetrahedrization.

- **Rule3**: If a tetrahedron inside $P$ has $R/L > \rho_0$, then split the tetrahedron by adding the circumcentre $x$ as a new vertex to the Delaunay tetrahedrization.

- Property: **Rule1** > **Rule2** > **Rule3**
Local density

- **Local feature size**: \( f : \mathbb{R}^3 \to \mathbb{R} \), \( f(x) \) varies slowly with \( x \)
- **Insertion radius**: \( r_{x} \) length of the shortest Delaunay edge with endpoint \( x \) immediately after adding \( x \)

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**Figure 3.14**: The insertion radius of a vertex \( v \) is the distance to the nearest vertex when \( v \) first appears in the mesh. (a) If \( v \) is an input vertex, \( r_{v} \) is the distance to the nearest other input vertex. (b) If \( v \) is the midpoint of a subsegment encroached upon by a mesh vertex, \( r_{v} \) is the distance to that vertex. (c) If \( v \) is the midpoint of a subsegment encroached upon only by a rejected vertex, \( r_{v} \) is the radius of the subsegment’s diametral circle. (d) If \( v \) is the circumcenter of a skinny triangle, \( r_{v} \) is the radius of the circumcircle.
Radii andParents

- Parent vertex: $p(v)$ is the vertex that is “responsible” for the insertion of $v$
Radius claims

- Let $x$ be a vertex of $D$ and $p$ is its parents, if it exits. Then $r_x \geq f(x)$ or $r_x \geq c \cdot r_p$ where $c = 1/\sqrt{2}$ if $x$ has type 1 or 2 and $c = \rho_0$.
Graded meshes

- Ratio claim: Let $x$ be a Delaunay vertex with parent and $r_x \geq c.r_p$ Then 
  \[ f(x) / r_x \leq 1 + f(p) / (c.r_p) \]

- Invariant: if $x$ is a type I vertex in the Delaunay tetrahedrization, for $i=1..3$ then 
  \[ r_x \geq f(x) / C_i \]

- Conclusion: 
  \[ \|x - y\| \geq f(x) / (1 + C_1) \]
Silver exudation

- Silver exits frequently between well shaped tetrahedral inside Delaunay tetrahedration

Figure 4.2: A silver tetrahedron.
Variational Tetrahedral Meshing

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Goal

- Tetrahedral mesh generation
- Focus on:
  - quality: shape of elements
  - control over sizing
    - dictated by simulation
    - constrained by boundary
    - low number of elements
Quality Metrics

- 3D:
  - Radius-edge ratio \( \rho = \rho(\sigma) = \frac{R}{L} \), \( R \): outer radius, \( L \): shortest edge
  - Aspect ratio \( \vartheta = \vartheta(\sigma) = \frac{R}{r} \), \( r \): inner radius
  - Volume ratio \( \varphi(\sigma) = \frac{V}{L^3} \), \( V \): volume

In the paper use \textit{Radius ratio} = \textit{Aspect ratio} – “fair” to measure silver
Popular Meshing Approaches

- Advancing front
- Specific subdivision
  - octree
  - crystalline lattice
- Delaunay
  - refinement
  - sphere packing

**Optimization:**
- spring energy
- aspect ratios
- dihedral angles
- solid angles
- volumes
- edge lengths
- containing sphere radii
- etc.

[Freitag et al.; Amenta et al.]
Delaunay Triangulation

- Degree of freedom: vertex positions
Optimizing Vertex Placement

Improve compactness of Voronoi cells by minimizing

\[ E = \sum_{j=1..k} \int_{x \in R_j} \rho(x) \left\| x - x_j \right\|^2 dx \]

Necessary condition for optimality: \textbf{Centroidal Voronoi Tessellation (CVT)}
Optimizing Vertex Placement

- **CVT [Du-Wang 03]**
  - best PL approximant
  - compact Voronoi cells
  - isotropic sampling

\[ E_{CVT} = \sum_{i=1}^{N} \int_{V_i} \|x - x_i\|^2 \, dx \]

*Vi: local cell*

Best L1 approximation of paraboloid
Alas, Harder in 3D...

- well-spaced points generate only round or sliver Delaunay tetrahedra [Eppstein 01]
Alas, Harder in 3D...

- well-spaced points generate only round or sliver Delaunay tetrahedra
- regular tetrahedron does not tile space
  \[
  (360° / 70.53° = 5.1)
  \]

\[\text{dihedral angle of the regular tetrahedron} = \arccos\left(\frac{1}{3}\right) \sim 70.53°\]
Idea: Underlaid vs Overlaid

CVT
- best PL approximant
- compact Voronoi cells
- isotropic sampling

$$E_{CVT} = \sum_{i=1..N} \int_{V_i} \|x - x_i\|^2 dx$$

ODT [Chen 04]
- best PL interpolant
- compact simplices – not dual
- isotropic meshing

$$E_{ODT} = \frac{1}{n+1} \sum_{i=1..N} \int_{\Omega_i} \|x - x_i\|^2 dx$$

Best L1 approximation of paraboloid
Figure 4: Nomenclature: Left: We denote by $\Omega_i$ the 1-ring of vertex $x_i$. Middle: $V_i$ is the Voronoi cell of vertex $x_i$. Right: The center of the circumcircle of triangle $T_j$, is denoted $c_j$, while its radius is denoted $R_j$. 
Rationale Behind ODT

- Approximation theory:
  - linear interpolation: optimal shape of an element related to the Hessian of $f$ [Shewchuk]

- $\text{Hessian}(\|x\|^2) = \text{Id}$
  - regular tetrahedron best
Which *PL interpolating mesh* best approximates the paraboloid?

- for **fixed vertex locations**
  - Delaunay triangulation is *the* optimal connectivity

- for **fixed connectivity**
  - min of quadratic energy leads to *the* optimal vertex locations
  - closed form as function of neighboring vertices
Optimizing Connectivity

- Delaunay triangulation is \textit{the} optimal connectivity
- Optimal connectivities minimize $E_{ODT}$
Optimizing Vertex Position

$$E_{ODT} = \frac{1}{4} \sum_i x_i^2 |\Omega_i| - \int_{\mathcal{M}} x^2 dx,$$

Simple derivation in $x_i$ lead to optimal position in its 1-ring

$$x_i^* = -\frac{1}{2 |\Omega_i|} \sum_{T_j \in \Omega_i} \left( \nabla x_i |T_j| \left[ \sum_{x_k \in T_j, x_k \neq x_i} ||x_k||^2 \right] \right).$$

$\nabla x_i |T_j|$ is the gradient of the volume of the tet $T_j$

Update vertex position

$$x_i^* = x_i - \frac{1}{2 |\Omega_i|} \sum_{T_j \in \Omega_i} \left( \nabla x_i |T_j| \left[ \sum_{x_k \in T_j} ||x_i - x_k||^2 \right] \right)$$
Optimal Update Rule

\[ x_i^* = \frac{1}{|\Omega_i|} \sum_{\tau_j \in \Omega_i} |\tau_j| c_j \]

circumcenter
Optimization

Alternate updates of

- connectivity (Delaunay triangulation)
- vertex locations
Optimization: Init

distribution of radius ratios

good
Optimization: Step 1

distribution of radius ratios

good
Optimization: Step 2

distribution of radius ratios

bad
good
Optimization: Step 50

distribution of radius ratios

good
Optimization: Step 50
Sizing Field

Goal reminder:
- minimize number of elements
- better approximate the boundary
- while preserving good shape of elements

Those are not independent!
- well-shaped elements iff K-Lipschitz sizing field [Ruppert, Miller et al.]
Automatic Sizing Field

Properties:
- size $\leq lfs$ (local feature size) on boundary
- sizing field = maximal K-Lipschitz

$$\mu (x) = \inf_{y \in \partial \Omega} \left[ K \| x - y \| + lfs(y) \right]$$

Parameter
Sizing Field: Example
3D Example

\[ x_i^* = \frac{1}{|\Omega|} \sum_{\tau_j \in \Omega_i} |\tau_j| \cdot C_j \]

local feature size        sizing field
Algorithm

Read the input boundary mesh $\partial \Omega$
Setup Data Structure & Preprocessing
Compute sizing field $\mu$
Generate initial sites $\mathbf{x}_i$ inside $\Omega$
Do

Construct Delaunay triangulation($\{\mathbf{x}_i\}$)
Move sites $\mathbf{x}_i$ to their optimal positions $\mathbf{x}_i^*$

Until (convergence or stopping criterion)
Extract interior mesh
Result
Stanford Bunny
Hand

local feature size
Hand: Radius Ratios
Torso
(courtesy A.Olivier-Mangon & G.Drettakis)
Fandisk
Comparison with the Unit Mesh Approach
Conclusion

- Generate high quality isotropic tetrahedral meshes (improved aspect ratios)
- Simple alternated optimizations
  - connectivity: Delaunay
  - vertex positions: weighted circumcenters
  - Good in practice
- Limit
  - Approximate input boundary instead of conforming
  - Theoretical guarantees to be developed
Thank you for your attention