Mixed aleatory and epistemic uncertainty quantification using fuzzy set theory

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A R T I C L E   I N F O

Article history:
Received 17 April 2015
Received in revised form 18 June 2015
Accepted 6 July 2015
Available online 22 July 2015

Keywords:
Aleatory uncertainty
Epistemic uncertainty
Fuzzy set
Uncertainty modeling
Marching cubes algorithm
Isocontour extraction

A B S T R A C T

This paper proposes algorithms to construct fuzzy probabilities to represent or model the mixed aleatory and epistemic uncertainty in a limited-size ensemble. Specifically, we discuss the possible requirements for the fuzzy probabilities in order to model the mixed types of uncertainty, and propose algorithms to construct fuzzy probabilities for both independent and dependent datasets. The effectiveness of the proposed algorithms is demonstrated using one-dimensional and high-dimensional examples. After that, we apply the proposed uncertainty representation technique to isocontour extraction, and demonstrate its applicability using examples with both structured and unstructured meshes.

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1. Introduction

Numerical modeling and simulations help to study and predict the physical events or the behavior of engineered systems [1], and also reduce the need for costly or cumbersome physical experiments [2]. However, due to the inherent variability of the physical systems, the assumptions embodied in the mathematical models, nonspecified physical characteristics, the numerical approximations, etc., uncertainty is inevitable in the modeling and simulation process [3]. Therefore, to provide useful and reliable information regarding the physical system, understanding and quantifying the uncertainty in the simulations become critical.

Uncertainty in simulations, according to its various sources, can be broadly categorized into two types: aleatory and epistemic. Aleatory uncertainty (stochastic or irreducible uncertainty) describes the variability in the physical system, and variables with aleatory uncertainty can be treated as random variables. Epistemic uncertainty, known as systematic or reducible uncertainty, is considered to be a consequence of the incomplete or inadequate knowledge of the physical system [4].

Probability theory (or the concept of probability theory) is well developed and has been considered as the most suitable choice for aleatory uncertainty quantification. While alternative mathematical frameworks, such as fuzzy set theory [5,6], possibility theory [7,8], evidence theory [9,10], fuzzy measures as a unifying structure [11], and generalized p-boxes [12], are explored for possible better representations of epistemic uncertainty. Recently, it is becoming more common that prob-

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http://dx.doi.org/10.1016/j.ijar.2015.07.002
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ability theory and other modern mathematical frameworks are combined together for uncertainty quantification due to the simultaneously presence of both aleatory and epistemic uncertainties in systems [3,13–17]. For example, Roy et al. implemented the mixed types of uncertainty propagation using probability theory coupled with interval analysis, and built a p-box to represent the propagated uncertainty in the model output [3]. Tang et al. coupled Dempster–Shafer theory and probability theory to quantify the mixed aleatory and epistemic uncertainty using a polynomial surrogate, and introduced local sensitivity analysis based on Dempster–Shafer theory [13]. Huang et al. dealt with the mixed types of uncertainty using both random variables and fuzzy variables in reliability based design optimization [16,18]. However, all the mentioned studies assume the mathematical representation for the uncertain input parameters, representing the mixed types of uncertainty in a given piece of information using different mathematical frameworks (i.e., modeling knowledge about uncertain input variables) is still under development.

In the current work, we focus on developing approaches for representing mixed types of uncertainty. Specifically, we consider the following scenario: $Y$ is a random variable with unknown PDF and its associated piece of information is in the format of a finite size of an ensemble (i.e., a finite number of samples). In such a situation, both aleatory and epistemic uncertainties exist. To represent the mixed types of uncertainty mathematically, we adopt the relatively well-developed nonprobabilistic theory – fuzzy set theory – to construct fuzzy probabilities.

The concept of fuzzy probabilities (defining a possibility measure/fuzzy set over a probability value) has been investigated in the literature [19–21]. For example, Zadeh was the first to propose “fuzzy probabilities” where probabilities are assumed to be fuzzy rather than real numbers [19]. They were defined in the situation where the propositions are represented by fuzzy sets. Buckley [20] has discussed the construction of “fuzzy probabilities” using nested confidence intervals and analyzed their properties from a theoretical point of view. Baudrit and Dubois [21] also mentioned the construction of possibility distribution based on nested confidence intervals for a presentation of incomplete probabilistic knowledge. In the current work, we propose simple methods for constructing fuzzy probabilities specifically aimed at mixed types of uncertainty representation and discuss the requirements of fuzzy probabilities in the field of uncertainty quantification. The main contributions of this study can be summarized as:

- Propose an uncertainty representation technique for modeling mixed types of uncertainty using the concept of fuzzy probabilities.
- Discuss the requirements of fuzzy probabilities for uncertainty representation in an ensemble.
- Demonstrate the properties of proposed uncertainty representation technique in both one-dimensional and high-dimensional examples.
- Apply the proposed uncertainty representation technique to isocontour extraction.

The remainder of the paper proceeds as follows. In Section 2, we provide a brief introduction to fuzzy set theory, where we introduce the membership function, the extension principle. Section 3 is devoted to the introduction of an uncertainty representation technique based on fuzzy probability, where we state the problem, and propose the requirements and algorithms for constructing fuzzy probabilities to model mixed types of uncertainty. We then demonstrate the properties of the proposed uncertainty representation technique using one-dimensional and high-dimensional examples in Section 4. In Section 5, we apply the proposed uncertainty representation technique to isocontour extraction; specifically, we extend the marching cubes algorithm to the fuzzy probabilistic marching cubes algorithm and demonstrate its effectiveness on a few examples. We summarize and conclude our work in Section 6.

2. Basics of fuzzy set theory

Fuzzy sets can be considered as a generalization of classical sets. The change of membership of each element in a fuzzy set is gradual and represented using a membership function.

Definition 1. Let $U$ be a classical nonempty set and $x \in U$ be an element. A fuzzy set $\tilde{A} \subset U$ is defined as

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x))| x \in U\},$$

(1)

where $\mu_{\tilde{A}}(x) : U \rightarrow [0, 1]$ is called the membership function of $\tilde{A}$.

The membership function $\mu_{\tilde{A}}(x)$ ( $\mu_{\tilde{A}}$ for simplicity) represents the degree of the membership of the element $x$ in the fuzzy set $\tilde{A}$. The element $x$ is considered as a full member of the fuzzy set $\tilde{A}$ if $\mu_{\tilde{A}}(x) = 1$; whereas $x$ is considered as not a member of the fuzzy set $\tilde{A}$ if $\mu_{\tilde{A}}(x) = 0$. A fuzzy set $\tilde{A}$ is called normal when $\sup_{x \in U} \mu_{\tilde{A}}(x) = 1$.

A fuzzy set can be characterized using classical sets, such as support and $\alpha$-cuts.

Support: \[ \text{Supp}(\tilde{A}) = \{x| \mu_{\tilde{A}}(x) > 0\}, \]

(2)

$\alpha$-cut: \[ [\tilde{A}]^{\alpha} = \{x| \mu_{\tilde{A}}(x) \geq \alpha\}. \]

(3)
The support of a fuzzy set \( \tilde{A} \) is a classical set whose elements have nonzero membership grade in \( \tilde{A} \). The \( \alpha \)-cut of a fuzzy set, also called \( \alpha \)-level, consists of elements that belong to a fuzzy set \( \tilde{A} \) with at least \( \alpha \) degree.

A function of crisp numbers can be extended to the function of fuzzy sets using Zadeh’s extension principle, in other words, a classical map \( f : U \rightarrow Y \) operating on elements \( x \in U \) can be extended towards a map \( \tilde{f} : \mathcal{F}(U) \rightarrow \mathcal{F}(Y) \), operating on fuzzy sets \( \tilde{A} \in \mathcal{F}(U) \), where \( \mathcal{F}(U) \) and \( \mathcal{F}(Y) \) are the collections of all the fuzzy sets defined on \( U \) and \( Y \), respectively.

**Definition 2.** Let \( \tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_M \) be \( M \) fuzzy sets with corresponding membership functions \( \mu_{\tilde{A}_1}, \mu_{\tilde{A}_2}, \ldots, \mu_{\tilde{A}_M} \) defined on \( U_1, U_2, \ldots, U_M \), respectively, and \( U \) be the Cartesian product \( U = U_1 \times U_2 \times \cdots \times U_M \). If \( f \) is a map from \( U \) to \( Y \), i.e., \( y = f(x_1, x_2, \ldots, x_M) \), then the extension principle allows us to define a fuzzy set \( \tilde{B} \) in \( \mathcal{F}(Y) \) by [22]

\[
\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) | y = f(x_1, x_2, \ldots, x_M), (x_1, x_2, \ldots, x_M) \in U \},
\]

where

\[
\mu_{\tilde{B}}(y) = \begin{cases} 
\sup_{(x_1, \ldots, x_M) \in f^{-1}(y)} \min\{\mu_{\tilde{A}_1}(x_1), \ldots, \mu_{\tilde{A}_M}(x_M)\} & \text{if } f^{-1}(y) \neq \emptyset, \\
0 & \text{if } f^{-1}(y) = \emptyset.
\end{cases}
\]

(4)

where \( f^{-1} \) is the inverse of \( f \).

For \( M = 1 \), the extension principle reduces to

\[
\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) | y = f(x), x \in U \},
\]

where

\[
\mu_{\tilde{B}}(y) = \begin{cases} 
\sup_{x \in f^{-1}(y)} \mu_{\tilde{A}}(x) & \text{if } f^{-1}(y) \neq \emptyset, \\
0 & \text{if } f^{-1}(y) = \emptyset.
\end{cases}
\]

(5)

3. Mixed types of uncertainty representation

3.1. Problem setup

Uncertainty representation is to create some structure for an uncertain variable associated with a piece of information mathematically. It is a modeling process. The piece of information could be in any form, such as data format or linguistic format. In the current work, we focus on uncertainty representation in an ensemble dataset \( \{y_i\}_{i=1}^N \) (where \( i \) is index) associated with a random variable \( Y \), whose probability density function is unknown to us. If there are infinitely many samples available for the random variable, one can construct a probability density function accurately. However, in practice, we may have only a limited number of samples and it consequently would be difficult, if not impossible, to obtain an accurate probability for any proposition \( A \) (e.g., \( A = \{Y|Y < \theta\} \), where \( \theta \) is a fixed value). In such a situation, it is suggested to represent the random variable \( Y \) “as a mixture of natural variability (aleatory) and estimation errors (epistemic)” since “a finite number of samples from a population leads to epistemic uncertainty [23].” We consider such a situation in the current work and build a fuzzy set for a probability measure: using a probability measure to represent the random nature in the quantity of interest \( Y \); and a fuzzy set to represent the epistemic uncertainty (due to the incompleteness of the information) in the probability measure. In the following, we adopt Zadeh’s concept of “fuzzy probability” and define it mathematically to represent both aleatory and epistemic uncertainty.

Let \( (X, \Omega, P) \) be a classical probability space, where \( X \) is a sample space, \( \Omega \) is a collection of events, and \( P \) is a probability measure. Then the fuzzy set \( \tilde{P}_N^A = \{(p_A, \mu_{\tilde{P}_N^A}) | p_A \in [0, 1]\} \) (where \( N \) is the number of samples available for the variable \( Y \), \( A \) is any proposition, \( \tilde{P}_N^A \) is used interchangeably with \( \tilde{P}_N^A \) for simplicity, \( p_A \) is the element in \([0, 1]\), and \( \mu_{\tilde{P}_N^A} \) is the membership function) is defined as the fuzzy probability of \( A \) if it satisfies

1. **Requirement R1:**

\[
\tilde{P}_N^A \rightharpoonup P(A) \quad \text{as} \quad N \rightarrow \infty.
\]

(6)

The requirement R1 can be considered as a weaker form of Hacking’s Frequency Principle, which originally states: when the objective probability of an event \( A \) is \( P(A) \), then the subjective probability of \( A \) is \( P(A) \) [24,25]. Accustomed to fuzzy probabilities, \( P_A \) should asymptotically become identical to \( P(A) \) when an infinite number of samples are available.

2. **Requirement R2:** \(|\text{Supp}(\tilde{P}_N^A)|\) decreases (with tolerance of oscillations) as \( N \) increases, where

\[
|\text{Supp}(\tilde{P}_N^A)| = \max\{|\text{Supp}(\tilde{P}_N^A)| - \min\{|\text{Supp}(\tilde{P}_N^A)|\}\}.
\]

As mentioned earlier, the fuzzy set represents the epistemic uncertainty in the probability of a proposition \( A \). As the sample size \( N \) increases (i.e., the knowledge regarding the chance of the occurrence of \( A \) increases), the epistemic
uncertainty should decrease, and one indication of this decrease is that the support of the fuzzy set shrinks. The oscillations are tolerated in this requirement since only a large number of samples can represent the true properties of a random variable. One may not observe the decreasing feature when \( N \) increases by one or two.

We would like that the constructed fuzzy sets satisfy the requirement \( R2 \), however, we can only prove it asymptotically. Therefore we relax the requirement \( R2 \) to be in the asymptotic sense for now.

To extend the additivity of the probability measure (i.e., \( P(\bar{A}) = 1 - P(A) \), where \( \bar{A} \) is the complement of \( A \)) to the fuzzy probability, we define the fuzzy probability of \( \bar{A} \) (denoted as \( \bar{P}_N^A \) or \( \bar{P}_N^A \) for simplicity) as follows.

**Definition 3.** Let the fuzzy probability of \( A \) (i.e., the fuzzy set for the probability of \( A \) ) constructed from \( N \) samples be \( \bar{P}_N^A \), then the fuzzy probability of \( \bar{A} \) is defined as

\[
\bar{P}_N^A = 1 - P_N^A, \quad \forall A \in \Omega.
\]

(7)

i.e., the support of the fuzzy set \( \bar{P}_N^A \) is

\[
\text{Supp}(\bar{P}_N^A) = [1 - \max\{\text{Supp}(\bar{P}_N^A)\}, 1 - \min\{\text{Supp}(\bar{P}_N^A)\}],
\]

(8)

and the degree of membership for \( \forall p_A \in \text{Supp}(\bar{P}_N^A) \) is

\[
\mu_{\bar{P}_N^A}(p_A) = \mu_{\bar{P}_N^A}(1 - p_A).
\]

(9)

An example of the membership functions of fuzzy sets \( \bar{P}_N^A \) and \( \bar{P}_N^A \) is shown in Fig. 1.

### 3.2. Construction of fuzzy probability in 1D case

Consider the proposition \( A = \{Y \geq \theta\} \) where \( \theta \) is a fixed deterministic value. To represent the mixed types of uncertainty in an ensemble \( \{y_i\}_{i=1}^N \) ( \( i \) is index), we construct the fuzzy probability \( \bar{P}_N^A \) to indicate the occurrence of proposition \( A \) i.e., we construct a membership function \( \mu_{\bar{P}_N^A} \) for the classical probability measure \( P(A) \) since it is uncertain to us due to the limited information. The membership function is constructed as follows,

\[
\mu_{\bar{P}_N^A}(p) = \begin{cases} 
1 - \frac{(p - p_A,)}{p_A - p_A,} & \text{if } p_A, \leq p \leq p_A^+ \\
\frac{(p - p_A,)}{p_A, - p_A,} & \text{if } p_A^+ \leq p < p_A, \\
0, & \text{otherwise}
\end{cases}
\]

(10)

where

\[
p_A, = n/N,
\]

(11)

where \( n \) is the number of ensemble data points belonging to the set \( A \), \( N \) is the size of ensemble, and \( p_A^+(p_A,) \) is the upper (lower) bound of the probability of \( A \), which is estimated using the simultaneous confidence interval proposed by Quesenberry et al. [26] and Goodman [25] as follows,
Proof. We have

\[
p_A^- = \frac{a + 2n - \sqrt{D}}{2(N + a)}, \quad (12) \\
p_A^+ = \frac{a + 2n + \sqrt{D}}{2(N + a)}, \quad (13)
\]

where \(a\) is the quartile of order \(1 - \beta\) of the chi-square distribution with one degree of freedom, and \(D = a(a + 4n(N - n)/N)\). The calculated \(p_A^-\) and \(p_A^+\) satisfy \(P(p_A^- \leq P(A) \leq p_A^+) \geq 1 - \beta\), i.e., the probability of “the probability of \(A\) is bounded by \(p_A^-\) and \(p_A^+\)” is no less than \(1 - \beta\). In the current work, we take one of the usual probability levels, \(\beta = 0.05\). The algorithm (labeled Algorithm 1) is given as follows.

Data: Ensemble dataset \(|Y|_{n=1}^N\)
Result: \(\hat{P}_A = \{(p_A, \mu_A)\}\) for any \(A \in \Omega\)
1. Evaluate the sample probability \(P_{A\alpha n} = n/N\) using Eq. (11).
2. Calculate the lower and upper bounds of the support, i.e., \(p_A^-\) and \(p_A^+\) using Eqs. (12) and (13).
3. Obtain the membership function \(\mu_A(p_A)\) using Eq. (10).

Algorithm 1: Construction of fuzzy probability for 1D dataset.

Theorem 1. The fuzzy set \(\hat{P}_A\) calculated using Algorithm 1 is a fuzzy probability of \(A\) satisfying the requirements R1 and R2.

Proof. We have

\[
\frac{\sqrt{D}}{2(N + a)} = \sqrt{\frac{a^2 + 4an(N - n)/N}{4(N + a)^2}} \\
\leq \sqrt{\frac{a^2}{4(N + a)^2} + \frac{4an(N - n)}{4N(N + a)^2}} \\
\to 0, \text{ as } N \to \infty.
\]

1. To prove \(\hat{P}_N^A \overset{p}{\to} P(A)\) as \(N \to \infty\) for \(A = [Y \geq \theta]\), it is equivalent to prove \(p_A^- \overset{p}{\to} P(A)\), \(p_A^+ \overset{p}{\to} P(A)\) as \(N \to \infty\).

Since

\[
p_A^- = \frac{a + 2n}{2(N + a)} - \frac{\sqrt{D}}{2(N + a)},
\]

and

\[
\frac{a + 2n}{2(N + a)} \to n/N, \quad \text{and} \quad \frac{\sqrt{D}}{2(N + a)} \to 0, \quad \text{as } N \to \infty,
\]

we have

\[
p_A^- \to n/N, \quad \text{as } N \to \infty.
\]

According to the weak law of large numbers (i.e., \(n/N \overset{p}{\to} P(A)\)), we have

\[
p_A^+ \overset{p}{\to} P(A), \quad \text{as } N \to \infty
\]

Likewise, we can show \(p_A^- \overset{p}{\to} P(A)\) as \(N \to \infty\).

2. \(|\text{Supp}(\hat{P}_N^A)| = |p_A^- - p_A^+| = \sqrt{D}/(N + a)\), since \(\sqrt{D}/(N + a) \to 0\) as \(N \to \infty\), we have \(|\text{Supp}(\hat{P}_N^A)| \to 0\) as \(N \to \infty\). Combine with \(|\text{Supp}(\hat{P}_N^A)| > 0\), we can conclude that \(|\text{Supp}(\hat{P}_N^A)|\) decreases (with tolerance of oscillations) as \(N\) increases asymptotically. \(\Box\)

Theorem 2. Let \(\hat{P}_A^1\) be the fuzzy probability of \(\hat{A}\) constructed using Eq. (10) with \(A\) replaced by \(\hat{A}\), and \(\hat{P}_A^2\) be the one calculated using Eq. (7), then we have \(\hat{P}_A^1 = \hat{P}_A^2\).

Proof. Since \(\hat{P}_A^2 = 1 - \hat{P}_A\), it is equivalent to prove

\[
\hat{P}_A^1 = 1 - \hat{P}_A.
\]
According to the extension principle, it is equivalent to prove for any \( p_A \in \text{Supp}(\hat{P}_A) \),

\[
1 - p_A \in \text{Supp}(\hat{p}_A^1) \quad \text{and} \quad \mu_{\hat{p}_A}(p_A) = \mu_{\hat{p}_A^1}(1 - p_A). \tag{15}
\]

First let us prove

\[
1 - p_A \in \text{Supp}(\hat{p}_A^1) \quad \forall p_A \in \text{Supp}(\hat{P}_A). \tag{16}
\]

The lower and upper bounds (\( p_{A^-} \) and \( p_{A^+} \)) of the support of \( \hat{P}_A \) are calculated from Eqs. (12) and (13); and lower and upper bounds (\( p_{A^-} \) and \( p_{A^+} \)) of the support of \( \hat{p}_A^1 \) are

\[
p_{A^-} = \frac{a + 2(N - n) - \sqrt{D}}{2(N + a)}, \quad p_{A^+} = \frac{a + 2(N - n) + \sqrt{D}}{2(N + a)}.
\]

In the following, we show that \( 1 - p_A \in [p_{A^-}, p_{A^+}] \) holds.

\[
p_A \in [p_{A^-}, p_{A^+}] \quad \Rightarrow \quad 1 - p_A \in [1 - p_{A^-}, 1 - p_{A^+}],
\]

\[
1 - p_A \in \left[ 1 - \frac{a + 2n + \sqrt{D}}{2(N + a)}, 1 - \frac{a + 2n - \sqrt{D}}{2(N + a)} \right],
\]

\[
1 - p_A \in \left[ \frac{a + 2(N - n) - \sqrt{D}}{2(N + a)}, \frac{a + 2(N - n) + \sqrt{D}}{2(N + a)} \right],
\]

\[
1 - p_A \in [p_{A^-}, p_{A^+}].
\]

The relation Eq. (16) is proved. Next we prove

\[
\mu_{\hat{p}_A}(p_A) = \mu_{\hat{p}_A^1}(1 - p_A). \tag{17}
\]

It is clear that Eq. (17) holds when \( p_A = p_{A,\text{est}} \). In the case of \( p_A < p_{A,\text{est}} \),

\[
\mu_{\hat{p}_A} = (p_A - p_A^-)/(p_{A,\text{est}} - p_A^-),
\]

\[
\mu_{\hat{p}_A^1} = (p_{A^+} - 1 - p_A)/(p_{A^+} - p_{A,\text{est}}).
\tag{18}
\]

Since

\[
p_{A,\text{est}} - p_A^- = \frac{n}{N} - \frac{a + 2n - \sqrt{D}}{2(N + a)} = \frac{2an - aN + \sqrt{DN}}{2(N + a)},
\]

\[
p_{A^+} - p_A^- = \frac{a + 2(N - n) + \sqrt{D}}{2(N + a)} - \frac{N - n}{N} = \frac{2an - aN + \sqrt{DN}}{2(N + a)},
\]

the equation \( p_{A,\text{est}} - p_A^- = p_{A^+} - p_{A,\text{est}} \) holds. Combined with \( p_A - p_A^- = p_{A^+} - (1 - p_A) \), which can be easily shown using simple algebraic operations, Eq. (17) holds for \( p_A < p_{A,\text{est}} \). Similarly, it can be proved for the case of \( p_A > p_{A,\text{est}} \).

### 3.3. Construction of fuzzy probability in high dimensions

Let the quantity of interest be a high-dimensional variable \( \vec{Y} = (Y^1, Y^2, \ldots, Y^M) \) (where \( 1, 2, \ldots, M \) are indices) and the proposition \( A = \{ Y \mid Y \subseteq A \} \), where \( A \) is the desired range. The available information corresponding to \( Y \) is an ensemble dataset \( \{ \vec{Y}^j \}_{j=1}^N \) with finite \( N \). In this section, we propose algorithms to construct the fuzzy probability \( \hat{P}_A \) to indicate the occurrence of the proposition \( A \), i.e., constructing a membership function \( \mu_{\hat{P}_A} \) for the classic probability measure \( P(A) \). In the following, two algorithms are proposed for two cases with and without an assumption of independence among \( Y^j \) (where \( j = 1, \ldots, M \) are indices).

#### 3.3.1. Independent case

Here, we assume \( Y^j \)'s are independent random variables, and \( A \) can be written as a Cartesian product \( A = A^1 \times \ldots \times A^M \) (where \( 1, \ldots, M \) are indices). Then the probability \( P(\{ Y^1, Y^2, \ldots, Y^M \} \in A) \) can be expressed as the multiplication of the probabilities for each component:

\[
P(\{ Y^1, Y^2, \ldots, Y^M \} \in A) = P(Y^1 \in A^1)P(Y^2 \in A^2) \ldots P(Y^M \in A^M),
\tag{19}
\]

where \( P(Y^j \in A^j) \) is uncertain due to the limited number of samples.
For each component $Y^j (j = 1, \ldots, M)$, we calculate the fuzzy probability $\tilde{P}^j_{A_i}$ to represent the uncertainty in $P(Y^j \in A^j)$ based on its corresponding $N$ samples $\{y^j_i\}_{i=1}^N$. Using Zadeh’s extension principle for the multiplication of fuzzy sets, the membership function of the fuzzy probability $\tilde{P}_A$ can be calculated to represent the uncertainty in $P(\{Y^1, Y^2, \ldots, Y^M\} \in A)$. The algorithm (labeled Algorithm 2) is

Data: Given $\{\tilde{y}^j_i\}_{i=1}^N$, where $\tilde{y} = \{y^j_i\}_{i=1}^M$ and $i, j$ are indices
Result: $\tilde{P}_A = \{(p_A, \mu_{p_A})\}$ for any $A \in \Omega$

while $j \leq M$

Evaluate $p_{A^j}, p_{A^j}^+$ and $p_{A^j}^-$ using Eqs. (11), (12) and (13) based on ensemble $\{y^j_i\}_{i=1}^N$.

Calculate the fuzzy probability $\tilde{P}^j_{A_i} = \{(p^j_{A_i}, \mu_{p^j_{A_i}})\}$ for $Y_j$ using Eq. (10).

end

Obtain $\tilde{P}_A = \{p_A, \mu_{p_A}\}$ using

$$
\mu_{p_A}(p_A) = \sup_{|p^j_{A_i} - p_{A^j}^+| \leq p_A} \min_{p_{A^j}^+ = p_A} (\mu_{p^j_{A_i}}(p^j_{A_i}), \ldots, \mu_{p_{A^j}^+} (p_{A^j}^+)),
$$

(20)

for any $p_A \in [0, 1]$.  

Algorithm 2: Construction of fuzzy probability for independent high-dimensional dataset.

Theorem 3. The fuzzy set $\tilde{P}_A$ calculated using Algorithm 2 is a fuzzy probability satisfying the requirements R1 and R2.

Proof.

1. Since $\tilde{P}^j_{A_i} \xrightarrow{P} P^j(A^j)$ for all $j$ as $N \to \infty$, $\tilde{P}_A \xrightarrow{P} P(A)$ as $N \to \infty$.
2. We have

$$
|\text{Supp}(\tilde{P}_A)| = \prod_j \max_k \text{Supp}(\tilde{P}^j_{A_i}) = \pi_j \min_k \text{Supp}(\tilde{P}^j_{A_i}),
$$

$$
\leq \sum_j |\text{Supp}(\tilde{P}^j_{A_i})|.
$$

We have shown that $|\text{Supp}(\tilde{P}^j_{A_i})| \to 0$ as $N \to \infty$, consequently, $|\text{Supp}(\tilde{P}_A)| \to 0$ as $N \to \infty$. Combined with $|\text{Supp}(\tilde{P}_A)| > 0$, one can conclude that $|\text{Supp}(\tilde{P}_A)|$ decreases (with tolerance of oscillations) as $N$ increases asymptotically. □

The fuzzy probability of $\tilde{A}$ can be directly calculated using Zadeh’s extension principle on $\tilde{P}_A = 1 - \sum_{j=1}^M \tilde{P}^j_{A_i}$ without calculating $\tilde{P}_A$.

3.3.2. Dependent or dependency-unknown case

Now we consider the situation where the components of high-dimensional quantity of interest $\tilde{Y} = \{Y^1, \ldots, Y^M\}$ are dependent or having unknown dependency. Assume we have an ensemble $\{\tilde{y}^j\}_{i=1}^N$ as the information and we consider the proposition $A = \{\tilde{y} \subseteq A\}$, where $A$ is the desired range. A fuzzy probability $\tilde{P}_A$ is used to represent the mixed types of uncertainty regarding the occurrence of the proposition. In such a situation, the membership function $\mu_{\tilde{P}_A}$ can be calculated using Algorithm 1, except $n$ is the number of $M$-dimensional ensemble data points $\tilde{y}^j$ falling in the set $A$, and $N$ is the size of the $M$-dimensional ensemble for $p_{A, \text{est}} = n/N$. We label it Algorithm 3. Clearly, the computational cost is independent of dimension.

Data: Given $\{\tilde{y}^j\}_{i=1}^N$ where $i$ is index
Result: $\tilde{P}_A = \{(p_A, \mu_{p_A})\}$ for any $A \in \Omega$

1: Evaluate the sample probability $p_{A, \text{est}} = n/N$, where $n$ is the number of $M$-dimensional ensemble data points $\tilde{y}^j$ falling in the set $A$.
2: Calculate the lower and upper bounds of the support, i.e., $p_A^-$ and $p_A^+$ using Eqs. (12) and (13).
3: Obtain the membership function $\mu_{p_A}(p_A)$ using Eq. (10).

Algorithm 3: Construction of fuzzy probability for dependent high-dimensional dataset.

For the purpose of distinction, we call the fuzzy probability constructed using Algorithm 3 for the dependent or dependency-unknown case $\tilde{P}_{\text{dep}}$, and the one using Algorithm 2 for the independent case $\tilde{P}_{\text{ind}}$.

Up to now, we have introduced simple methods of constructing fuzzy probabilities for both one-dimensional and high-dimensional cases, which satisfy the discussed requirements. Note that the construction methods based on nested
confidence intervals in the literature [20] also satisfy the requirements and can be extended for high-dimensional cases. However, we only focus on the proposed simple methods in the current work and demonstrate their effectiveness using examples and applications.

4. Examples

In this section, we demonstrate the effectiveness of the fuzzy probabilities in mixed aleatory and epistemic uncertainty representation and the proposed construction algorithms using both one-dimensional and high-dimensional examples.

4.1. Example in 1D case

Let $Y$ be a random variable with a set of samples $\{y^i\}_{i=1}^N$ (where $i$ is index), and $A = \{Y > 0\}$ be the proposition, and we consider the occurrence of $A$. The samples are drawn from a normal distribution with $N = 100, 200, 300, \ldots, 200000$. Since an infinite number of samples are not available, one cannot obtain a deterministic $P(A)$. We use a membership function in fuzzy set theory to quantify the epistemic uncertainty in the probability $P(A)$, i.e., construct a fuzzy probability $\tilde{P}_A$ to indicate the occurrence of the proposition $A$. The lower and upper bounds of the support of $\tilde{P}_A$ with respect to the number of samples are shown in Fig. 2(a). The result shows that the support (the interval from the lower bound $p_A^-$ to upper bound $p_A^+$) of the fuzzy probability of $A$ includes the true probability $P(A) = 0.5$, and the upper and lower bounds converge to 0.5 as the ensemble size goes to infinity, i.e., $N \rightarrow \infty$; consequently, the fuzzy probability of $A$ converges to $P(A)$ as $N \rightarrow \infty$. In addition, one can observe that as the number of samples increases, the length of the support of $\tilde{P}_A$ decreases (with the tolerance of oscillations), which is consistent with the fact that the epistemic uncertainty decreases as the information increases (i.e., the number of sample increases). The lower and upper bounds of the support of $\tilde{P}_A$ are shown in Fig. 2(b).

4.2. Examples in high-dimensional case

4.2.1. Independent dataset

Let $\{y^{1,i}\}_{i=1}^N, \ldots, \{y^{4,i}\}_{i=1}^N$ ($N = 30$) be samples drawn from normal distribution (with ‘seed’ 111 using MATLAB) independently. We consider the proposition

$$A = \{Y^1, Y^2, Y^3, Y^4|\exists 1 \leq j, k \leq 4, Y^j > 0 \text{ and } Y^k < 0\}.$$  

The complement of $A$ is $\bar{A} = \{Y^1, Y^2, Y^3, Y^4|Y^j > 0, \forall j\} \cup \{Y^1, Y^2, Y^3, Y^4|Y^j < 0, \forall j\}$. Using the additivity of a probability measure, we have

$$P(A) = 1 - \prod_{j=1}^4 P^j(Y^j > 0) - \prod_{j=1}^4 P^j(Y^j < 0),$$

$$= 1 - \prod_{j=1}^4 P^j(Y^j > 0) - \prod_{j=1}^4 (1 - P^j(Y^j > 0)).$$

Using Algorithm 2, one can construct the fuzzy probability $\tilde{P}\text{ind}_A$ to indicate the occurrence of the proposition $A$. For reference purposes, the true probability of $A$ is calculated as $P(A) = 1 - 0.5^4 - 0.5^4 = 0.8750$.

The membership function of the fuzzy probability $\tilde{P}^j_{A^1}$ (where $A^1 = \{Y^j > 0\}$ for each component $Y^j$ ($j = 1, \ldots, 4$ is index)) is calculated using Algorithm 1 and shown in Fig. 3(a). The membership function of $\tilde{P}\text{ind}_A$ is shown in Fig. 3(b). The membership function indicates that the probability with the highest degree of membership “1” is the same as the sample probability 0.8819. The true probability $P(A) = 0.8750$ is included in the support of the obtained fuzzy probability $\tilde{P}\text{ind}_A$ with a degree of membership larger than 0.9. We are also interested in the membership functions with respect to the number of samples, four membership functions with $N = 30, 60, 90, 120$ are shown in Fig. 3(c). Fig. 3(d) shows the lower
and upper bounds of the support of \( \tilde{P} \text{ind}_A \) with respect to the sample size \( N \). One can observe that the supports contain the true probability \( P(A) = 0.8750 \) as an element, and the lower and upper bounds of the supports converge to the true probability, consequently, the length of the support decreases as the sample size increases.

4.2.2. Dependent dataset

Let \( \{ \tilde{y}^i \}_{i=1}^{N} \) \((N = 100)\) be samples drawn from a normal distribution with a zero mean and covariance matrix

\[
\sigma = \begin{pmatrix}
1 & 0.5 & 0.3 & 0.1 \\
0.5 & 1 & 0.3 & 0.1 \\
0.3 & 0.3 & 1 & 0.1 \\
0.1 & 0.1 & 0.1 & 1
\end{pmatrix},
\] (22)

and we consider the proposition \( A \) defined as Eq. (21). For reference purposes, the estimated true probability of \( A \) using Monte Carlo with one billion samples is \( P(A) = 0.7567 \).

The membership function of \( \tilde{P} \text{dep}_A \) is shown in Fig. 4(a). As in the independent case, it indicates that the probability with the highest degree of membership "1" is the same as the sample probability. The true probability \( P(A) \) is included in the support of the fuzzy probability \( \tilde{P} \text{dep}_A \). The membership functions of the fuzzy probability of \( A \) with different numbers of samples \( N \) are shown in Fig. 4(b). Fig. 4(c) shows the length of the supports of the fuzzy probabilities with respect to \( N \). One can observe that the length of support decreases as \( N \) increases.

4.2.3. Comparison of algorithms for \( \tilde{P} \text{ind} \) and \( \tilde{P} \text{dep} \)

It is interesting to test the performance of two algorithms (Algorithm 2 and Algorithm 3) on both dependent and independent data with a large sample size \( N = 1000000 \). Figs. 5(a), (b) show (i) both \( \tilde{P} \text{ind}_A \) (the solid curve) and \( \tilde{P} \text{dep}_A \) (the dashed curve) give the correct results for an independent dataset as \( N \) is large; however, \( \tilde{P} \text{ind}_A \) has narrower support (which shows as a line in Fig. 5 due to the much faster convergence of the lower and upper bounds to the same value) compared to that of \( \tilde{P} \text{dep}_A \), which is due to the construction process; and (ii) \( \tilde{P} \text{dep}_A \) (the dashed curve) produces correct results in the sense that it approximates the true probability of \( A \) whereas \( \tilde{P} \text{ind}_A \) (the solid curve) does not for the correlated dataset (see Fig. 5(b)).

Fig. 5(c) shows the comparison of the results from the two algorithms for \( \tilde{P} \text{ind} \) and \( \tilde{P} \text{dep} \) on independent dataset with \( N = 100 \). One can observe both the supports of \( \tilde{P} \text{ind} \) and \( \tilde{P} \text{dep} \) include the true probability \( P(A) = 0.8750 \), however, \( \tilde{P} \text{ind} \) is more accurate since (i) the element with degree of membership 1 is closer to the true probability; (ii) the degree of
The fuzzy probability of A based on correlated Gaussian samples: (a) The membership function of \( \hat{P}_{\text{dep}} \) from the ensemble with size \( N = 100 \); (b) The membership functions of \( \hat{P}_{\text{dep}} \) with respect to ensemble size \( N = 100, 300, 600, 900 \); and (c) The length of the support of \( \hat{P}_{\text{dep}} \) with respect to the sample size \( N \).

The comparison of the fuzzy probabilities obtained from Algorithm 2 and Algorithm 3 (the solid curve for \( \hat{P}_{\text{ind}} \) and the dashed curve for \( \hat{P}_{\text{dep}} \)) for (a) independent dataset with \( N = 1000000 \); (b) dependent dataset with \( N = 1000000 \); and (c) independent dataset with \( N = 100 \).

In conclusion, Algorithm 2 (for \( \hat{P}_{\text{ind}} \)) is preferable for independent dataset whereas Algorithm 3 (for \( \hat{P}_{\text{dep}} \)) is more suitable for dependent dataset.

5. Applications to isocontour extraction

5.1. Conventional marching cubes algorithm

Visualization has become an integral component of data analysis tasks in a wide variety of applications ranging from scientific modeling and simulation to biomedical applications. One of the predominant data types, widely used in application, is scalar fields. Instead of the whole scalar fields, practitioners are oftentimes interested in studying specific levelsets of the scalar fields. For instance, meteorologists look at levelsets of temperature fields to issue warnings or in fluid simulation applications, the levelsets of pressure fields are used to determine the presence, size, and shape of vortex structures in a flow. Considering a multivariate function \( y = f(x) \in \mathbb{R}^d \), a levelset of such a function associated with isovalue \( \theta \) is defined as:

\[
C = \{ f(x) = \theta | x \in \mathbb{R}^d \}.
\]

In practice, the underlying function is often sampled on a uniform (for instance, multidimensional Cartesian) grid. Hence, visualization of the levelset of a multidimensional function requires approximation of the underlying function and then extraction of the levelset. Marching cubes is a simple algorithm developed to approximate and visualize the levelsets of multivariate functions (i.e., isocontours in 2D and iso-surfaces in 3D) [27].

For simplicity we explain the main idea of the marching cubes algorithm for a Cartesian grid in 3D but the idea can be easily modified for isocoutour extraction in 2D or any other structured or unstructured grids in two or three dimensions. The marching cubes algorithm proceeds through the scalar field and consider each cell of the scalar field (i.e., a cube for the Cartesian grid) one at a time. It uses trilinear interpolation (i.e., linear interpolation along the three major axes) to approximate the underlying multidimensional function in each cell. Trilinear approximation requires the sampling points only at the corners of a cell (i.e., a cube on a 3D Cartesian lattice). Therefore, given an isovalue \( \theta \) and the values of a scalar field at the corners of each cube constituting the scalar field, the presence of the iso-surface in that cell, which we call cell crossing, can be specified based solely on the values at the corners. A cell crossing happens if at least one sign change occurs in the set of differences \( \{ y^j - \theta \} \) (\( j \) is index of the corners of a cell) for the scalar values \( y^j \) at the corners of the cell in
question. In the presence of a cell crossing, the values at the corners of the cube also determine the spatial location of the (approximated) isocontour [27,28].

In the presence of uncertainty, the scalar field is no longer deterministic, and consequently the conventional marching cubes algorithm may not produce accurate information regarding the spatial location of the isocontour. To deal with the uncertainty in the scalar field (associated with ensemble data values at each grid location), a probabilistic version of the marching cubes algorithm was proposed [29]. The probabilistic marching cubes (PMC) algorithm uses random variables to model the uncertainty in the ensemble data. In each cell, based on the ensemble data \( Y_{1}^{N} \) (where \( N \) is the ensemble size, \( Y \) represents one realization at all corners), a multivariate Gaussian distribution is estimated for \( Y = Y_{1}^{M} \), where \( Y_{j} \) is the random variable for \( j \)-th corner of the cell. Following which, the PMC algorithm calculates the probability of the presence of the isocontour at each grid cell instead of extracting the exact location of the isocontour. The probability of the presence of the isocontour in a cell, i.e., the probability of cell crossing, is defined as

\[
P(\text{cell crossing}) = P(\bar{Y} | 31 \leq j, k \leq M, Y_{j}^{1} \geq \theta \text{ and } Y_{k}^{k} \leq \theta),
\]

where \( M \) is the number of corners at each cell.

However, the estimation of the multivariate distribution may not reflect the true distribution of \( \bar{Y} \) due to the finite number of the ensemble. In addition, random variables in probability theory may not always be the best choice to model the uncertainty in such a situation since both aleatory and epistemic uncertainties exist. In the following, we extend the PMC algorithm to the fuzzy probabilistic marching cubes algorithm and construct a fuzzy probability to indicate the occurrence of cell crossing.

5.2. Fuzzy probabilistic marching cubes algorithm

For computing the cell crossing fuzzy probabilities in 2D, we consider \( M = 4 \) random variables \( Y_{1}^{1}, Y_{2}^{2}, \ldots, Y_{M}^{M} \) at \( M \) grid points (the four corners of a cell). The variables are associated with finite \( M \)-dimensional ensemble data. Due to the presence of the mixed types of uncertainty, the deterministic probability of cell crossing (i.e., \( P(\text{cell crossing}) \) in Eq. (23)) may not be obtained. Instead, we construct a fuzzy set \( \tilde{P} \) to represent the epistemic uncertainty in the probability of cell crossing at each cell. In other words, we construct a fuzzy probability to indicate the presence of cell crossing using the proposed algorithms. Since ignoring the correlation among the ensemble for different corners may result in an incorrect isocontour [30], we use Algorithm 3 discussed in Section 3.3.2 (i.e., \( P_{\text{dep}} \)) to take into account the correlations.

Instead of extracting deterministic isocontours using the conventional marching cubes algorithm, fuzzy probabilistic marching cubes algorithm calculate the fuzzy probability which indicates the occurrence of cell crossing at each cell. Due to the presence of aleatory uncertainty in an ensemble, one cannot extract and visualize the exact deterministic position of the isocontour, the best one can do is to calculate and visualize the probability of the occurrence of the isocontour at each grid cell. In the situation where both epistemic and aleatory uncertainties exist, one cannot even obtain the deterministic probability of cell crossing accurately, instead we calculate the fuzzy probability of cell crossing and visualize its characteristics. In other words, it can be considered that the aleatory uncertainty is shown as multiple possible locations of isocontours with different degrees of probability (visualized in different colors); and the epistemic uncertainty is indicated by the difference between the lower and upper bounds of support of the fuzzy probability at each cell.

Note that one may not feel comfortable to make decisions based on fuzzy sets directly in applications. Therefore we adopt the concept of the mean value proposed in [31] to represent the characteristics of a fuzzy set and hope this concept can help decision-makers. Although there exists different definitions for the mean value in the literature [32], we leave the comparison of different choices for future research since the concept is only used in decision-making process, which is not the focus of the current work.

**Definition 4.** The mean of a fuzzy set \( \tilde{A} \) is defined as [31]

\[
E(\tilde{A}) = \frac{\int_{A=0}^{1} \alpha(a_{1}(\alpha) + a_{2}(\alpha))/2da}{\int_{A=0}^{1} \alpha da} = \frac{1}{\int_{A=0}^{1} \alpha da} \int_{A=0}^{1} \alpha(a_{1}(\alpha) + a_{2}(\alpha))da,
\]

where \( a_{1}(\alpha) \) and \( a_{2}(\alpha) \) are the lower and upper bounds of the \( \alpha \)-cut of the fuzzy set \( \tilde{A} \) (i.e., \( [\tilde{A}]^{\alpha} = [a_{1}(\alpha), a_{2}(\alpha)] \)).

5.3. Results and discussion

In this section, we use three examples to demonstrate the application of the fuzzy probabilities representing mixed types of uncertainty to the uncertain isocontours extraction. We will start with a synthetic example providing the lower and upper bounds of the support, and the mean of the fuzzy probability of cell crossing at each cell; and then demonstrate the utility of the proposed uncertainty representation technique for an application from computational fluid dynamics. We then implement the fuzzy probabilistic marching cubes algorithm on a 2D unstructured dataset to show that the algorithm is applicable to unstructured cases. In all the examples, the colormap has been scaled so that the highest value is assigned to red and the minimum value is assigned to white. This scaling of the colormap helps to provide a better color contrast.
5.3.1. Synthetic test dataset

The 2D (size $512 \times 512$) synthetic test dataset consists of an ensemble of size $30$. We use an implicit formulation of a diamond as $|x| + |y| \leq r$ and generate the ensemble by varying the value of $r$ according to $r \sim \mathcal{N}(2.0, 0.4^2)$. In order to construct a nontrivial and interesting example, each ensemble member has been contaminated with correlated random noise.

The fuzzy probability values over the whole field have been computed for an isovalue of $\theta = 0.5$. They quantify the mixed types of uncertainty in the spatial location of the isocontour, which is propagated from the uncertainty in the data. Visualization of fuzzy probabilities over the whole field would be problematic. Therefore we show only the lower/upper bounds of the supports of the fuzzy probabilities in Fig. 6(a) and (c). The lower/upper bounds of support of a fuzzy probability can be considered as the lower and upper bounds that bound the true probability. Fig. 6(a)/(c) indicates the region (in red or orangish-yellow) that has relatively higher lower/upper bounds of probability to have cell crossing. However, as mentioned before, it may not be straightforward (or one may not feel comfortable) to make decisions regarding the spatial location of an isocontour based on the comparison of the upper and lower bounds of the chance of cell crossing. Therefore, we also calculate the mean of the fuzzy probability of cell crossing (shown as Fig. 6(b)) for the purposes of decision-making.

We are also interested in comparing the result from the fuzzy probability to the true probability of cell crossing using the probabilistic marching cubes algorithm. To estimate the true probability of cell crossing, we construct the 2D synthetic dataset of an ensemble with size $1000$, and then calculate the ratio of the number of cell crossing to the total size $1000$ (visualized in Fig. 7(a)). The dataset of the ensemble of size $30$, which we use to demonstrate the proposed fuzzy probabilistic marching cubes algorithm and calculate the results shown in Fig. 6(a)-(c), is a subset of the constructed ensemble with size $1000$. From Fig. 6(b) and Fig. 7(a), one can easily observe that both the mean of the fuzzy probability and the true probability are relatively higher for the same region.

In addition, we compare the lower and upper bounds of the support to the true probability values. Fig. 7(b) shows the upper and lower bounds of supports and true probability values for the cells at the 250th row. From Fig. 7(b), one can tell that i) as expected, the upper and lower bounds of the support of fuzzy probability bound the true probability ii) the upper bounds are nonzero (equal to $p^+$) where the true probabilities are zero, which reflects the consideration of the proposed fuzzy probabilistic marching cubes algorithm on the epistemic uncertainty due to the limited size of the ensemble; iii) the lower and upper bounds of the supports have the same trend as the true probability.

---

Fig. 6. (a) The lower bounds of the supports of fuzzy probabilities of cell crossing; (b) The mean of fuzzy probabilities of cell crossing; (c) The upper bounds of the supports of fuzzy probabilities of cell crossing. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

Fig. 7. (a) The true probability of cell crossing, and (b) the comparison of lower (the curve in the bottom) and upper (the curve on the top) bounds of support to true probabilistic values (the curve in the middle).
5.3.2. Fluid mechanics example

Our second example is from the study of vortices in a computation fluid dynamics (CFD) application where scientists are interested in the size, number, and position of the vortices present in a flow field generated by the presence of an obstacle. Following the simplest approach, we study the pressure field (as a proxy for vorticity) in a flow field to investigate the vortex structures. The minimum pressure value usually corresponds to the center of a vortex. Therefore, one can use isocontours of the pressure field to estimate the position of the vortices in a flow field. In experiments, the behavior of vortices is usually uncertain due to the unavoidable uncertainty in the parameters such as Reynolds number, initial conditions, and boundary conditions. In order to mimic the uncertainty present in the parameters of a real fluid flow, we generated an ensemble of simulation runs with size 40 (see Fig. 8(a)) for the fluid flow scenario by random perturbation of the inlet velocity and the Reynolds number.

In this example, we used the 2D incompressible Navier–Stokes solver as part of the Nektar++ software package [33] to produce simulation results for fluid passing a stationary obstacle (in this case a cylinder). We performed a preprocessing step to normalize the pressure field of each ensemble member based on the average of the pressure value for a unique and fixed point inside the field behind the cylinder. After normalization of the pressure field for all ensemble members, we used isovalue $= -0.005$ for isocontour extraction.

Similar to our canonical example, we calculate the fuzzy probabilities of cell crossing for the pressure field of the flow and visualize the lower and upper bounds of supports in Fig. 8(b) and (d). The upper bound indicates the maximum chance of cell crossing at each cell and the region near the outflow boundary (in red or orangish-yellow) has a relatively higher maximum chance of cell crossing. For the purposes of decision-making, we also provide the mean of fuzzy probability of cell crossing. Fig. 8(c) shows that the region near the outflow boundary with yellow or light green color (besides the region around the obstacle) has a relatively higher mean of fuzzy probability of cell crossing.

In the following, we demonstrate on a triangular mesh example that the proposed fuzzy probabilistic marching cubes algorithm is applicable for unstructured mesh examples.

5.3.3. Unstructured mesh example

This example deals with the dataset from the study on an electrocardiographic forward problem [34]. The dataset consists of the electric potential field over a 2D torso slice (see Fig. 9(a) [34]) calculated using the finite element method with the triangular mesh and gPC collocation method [35]. There are 3859 vertices and 6893 triangles in the computational mesh (see Fig. 9(b)). The ensemble of potential field results from a single uniformly distributed random model parameter. Since the isovalue itself is not our interest, here we take isovalue $\theta = 0$ as an example.

The fuzzy probabilistic marching cubes algorithm is implemented to obtain the fuzzy probabilities of the occurrence of the isocontour at each cell (triangle). The obtained fuzzy probabilities model the mixed types of uncertainty in the spatial location of the isocontour. The fuzzy probability for each triangle is assigned to its center and the command TriScatteredInterp in Matlab is used to obtain the fuzzy probabilities for the whole torso slice for a better visualization. The lower and upper bounds of the supports, and the mean of fuzzy probabilities of cell crossing are shown in Fig. 10. The boundary curves are added for the reference of the location in the torso. The region in red indicates a high chance of cell crossing.

6. Summary and conclusions

Proper modeling and representation of mixed aleatory and epistemic uncertainty in a given piece of information is a challenging problem. In this paper, we use the well-developed probability theory to model the aleatory uncertainty. Due to the existence of epistemic uncertainty, the deterministic probability of any proposition may not be obtained. The uncertainty
in the probability of a proposition is represented by a fuzzy set. In other words, we propose algorithms to construct fuzzy probabilities to indicate the occurrence of propositions. Specifically, we discuss the requirements for the fuzzy probabilities in order to represent the mixed types of uncertainty, and propose algorithms to construct fuzzy probabilities for both independent and dependent datasets. The effectiveness of the proposed algorithms is demonstrated using one-dimensional and high-dimensional examples. The proposed uncertainty representation technique is applied to isocontour extraction, and we demonstrate its applicability using examples with both structured and unstructured meshes.

Acknowledgement

The second author acknowledges the support of National Science Foundation (NSF) grant IIS-1212806. The first and third authors acknowledge support by the Army Research Laboratory under Cooperative Agreement Number W911NF-12-2-0023. The views and conclusions contained in this document are those of the authors and should not be interpreted as representing the official policies, either expressed or implied, of the Army Research Laboratory or the U.S. Government. The U.S. Government is authorized to reproduce and distribute reprints for Government purposes not withstanding any copyright notation herein.

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