Compressed Sensing and OMP

Least Squares Regression

\[ A \mathbf{x} = \mathbf{b} \implies \mathbf{x} = (A^T A)^{-1} A^T \mathbf{b} \]

sparse if \( d > n \) \( \implies \) many solutions \( \mathbf{x} \) with 0 error

Compressed Sensing

\[ \text{m << n, non-zeros in} \ \mathbf{x} \]

[Tao - Candés] [Donahue] 2004

\[ (\text{model} \ \mathbf{x}) \quad S \ \text{signal, sparse} \quad |S| = d \]

\[ S = [0, 1, 0, 0, 0, 0, 0, \ldots, 0, 0] \]

Measurement matrix \( \mathbf{X} \) \( \text{谎言} \times \text{ fascist integers} \)

\[ y = \mathbf{X} S^T \] \( \text{actual measurements} \)

measurement row \( \mathbf{x}_i \)

\[ M \ll d \implies \] Recover S exactly

\[ M = K \cdot s \log \left( \frac{d}{s} \right) \]

\( K \in [4, 20] \)

\[ y_i = x_i^T S = (x_{i,1}, S) = (1, 0) + (-1, 0) + 0 + 0 + (1, 0) \]

\[ = 0 \]
Examples

- Single-pixel camera
  \[ 10 \text{ Gpixels} \rightarrow 2 \text{ Gpixels} \]
  
- Occulstion mask
  one row \( x_i \)

- Hubble Space Telescope

- MRI (on kids)

Given matrix \( X \), measurements \( y \)

How do we recover \( X \)?

Orthogonal Matching Pursuit (OMP)

- Choose column \( x_j \) of \( X \) most useful
  \[ x_j = \arg \max_{x_j \in X} \left| \langle y, x_j \rangle \right| \]

- \( y = \arg \min_{y \in \mathbb{R}} \| y - x_j y \| + \alpha \| y \| \Rightarrow S_j = x_j \)

- Residual \( r = y - x_j y \)

Repeat until \( r \) all-zeros, or \( \| r \| \) small enough.
$$c + \text{th round} \quad x_3^1, \ldots, x_3^{t+1}$$

$$\begin{bmatrix} Y \\ \eta \end{bmatrix} = \arg\min_{Y \in \mathbb{R}^t} \left\| Y - \left[ x_3^1, x_3^2, \ldots, x_3^{t+1} \right]^T \right\|_2^2 + \alpha \left\| \eta \right\|_2^2,$$

$$\eta = (x_3^3 \cdots x_3^{t+1})^\top x_3^{t+1} \eta$$