Sample Spaces, Events, Probability

CS 3130/ECE 3530:
Probability and Statistics for Engineers

August 28, 2014
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Examples:

- $A = \{3, 8, 31\}$
- $B = \{\text{apple, pear, orange, grape}\}$
- **Not** a valid set definition: $C = \{1, 2, 3, 4, 2\}$
Sets

Order in a set does not matter!

\[ \{1, 2, 3\} = \{3, 1, 2\} = \{1, 3, 2\} \]
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▶ When \( x \) is an element of \( A \), we denote this by:

\[ x \in A. \]
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- The “empty” or “null” set has no elements:
  \[ \emptyset = \{ \} \]
Definition

A **sample space** is the set of all possible outcomes of an experiment. We’ll denote a sample space as $\Omega$. 

Examples:

- **Coin flip:** $\Omega = \{H, T\}$
- **Roll a 6-sided die:** $\Omega = \{1, 2, 3, 4, 5, 6\}$
- **Pick a ball from a bucket of red/black balls:** $\Omega = \{R, B\}$
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Some Important Sets

- **Integers:** \( \mathbb{Z} = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \} \)
- **Natural Numbers:** \( \mathbb{N} = \{ 0, 1, 2, 3, \ldots \} \)
- **Real Numbers:** \( \mathbb{R} = \) "any number that can be written in decimal form"
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$5 \in \mathbb{R}, \quad 17.42 \in \mathbb{R}, \quad \pi = 3.14159\ldots \in \mathbb{R}$
Building Sets Using Conditionals

Alternate way to define natural numbers:
\[\mathbb{N} = \{ x \in \mathbb{Z} : x \geq 0 \} \]

Set of even integers:
\[\{ x \in \mathbb{Z} : x \text{ is divisible by } 2 \} \]

Rationals:
\[\mathbb{Q} = \{ p/q : p, q \in \mathbb{Z}, q \neq 0 \} \]
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Subsets

Definition
A set $A$ is a **subset** of another set $B$ if every element of $A$ is also an element of $B$, and we denote this as $A \subseteq B$. 

Examples:
- $\{1, 9\} \subseteq \{1, 3, 9, 11\}$
- $\mathbb{Q} \subseteq \mathbb{R}$
- $\{\text{apple, pear}\} \not\subseteq \{\text{apple, orange, banana}\}$
- $\emptyset \subseteq A$ for any set $A$
- $A \subseteq A$ for any set $A$ (but $A \not\subset A$)
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- You roll a die and get an even number: 
  \( \{2, 4, 6\} \subseteq \{1, 2, 3, 4, 5, 6\} \)
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- You roll a die and get an even number: 
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- You flip a coin and it comes up “heads”: 
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- Your code takes longer than 5 seconds to run:
  \[ (5, \infty) \subseteq \mathbb{R} \]
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$A = \{1, 3, 5\}$  \quad \text{“an odd roll”}

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Note: If $A \cap B = \emptyset$, we say $A$ and $B$ are **disjoint**.
Set Operations: Complement

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The **complement** of a set \( A \subseteq \Omega \), denoted \( A^c \), is the set of all elements in \( \Omega \) that are not in \( A \).
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A = \{1, 3, 5\} \quad \text{“an odd roll”}
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Set Operations: Difference

Definition

The **difference** of a set $A \subseteq \Omega$ and a set $B \subseteq \Omega$, denoted $A - B$, is the set of all elements in $\Omega$ that are in $A$ and are not in $B$.

Example:

$A = \{3, 4, 5, 6\}$

$B = \{3, 5\}$

$A - B = \{4, 6\}$

Note: $A - B = A \cap B^c$
DeMorgan’s Law

Complement of union or intersection:

\[(A \cup B)^c = A^c \cap B^c\]

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What is the English translation for both sides of the equations above?
Exercises

Check whether the following statements are true or false. (Hint: you might use Venn diagrams.)

- $A - B \subseteq A$
- $(A - B)^c = A^c \cup B$
- $A \cup B \subseteq B$
- $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
A probability function on a finite sample space $\Omega$ assigns every event $A \subseteq \Omega$ a number in $[0, 1]$, such that

1. $P(\Omega) = 1$
2. $P(A \cup B) = P(A) + P(B)$ when $A \cap B = \emptyset$

$P(A)$ is the probability that event $A$ occurs.
Equally Likely Outcomes

The number of elements in a set $A$ is denoted $|A|$. 

Example: Rolling a 6-sided die

$\text{\ } \quad P(\{1\}) = \frac{1}{6}$

$\text{\ } \quad P(\{1, 2, 3\}) = \frac{1}{2}$
Equally Likely Outcomes

The number of elements in a set $A$ is denoted $|A|$. If $\Omega$ has a finite number of elements, and each is equally likely, then the probability function is given by

$$P(A) = \frac{|A|}{|\Omega|}$$

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Repeated Experiments

If we do two runs of an experiment with sample space $\Omega$, then we get a new experiment with sample space

$$\Omega \times \Omega = \{(x, y) : x \in \Omega, y \in \Omega\}$$
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The element $(x, y) \in \Omega \times \Omega$ is called an ordered pair.
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Properties:
Order matters: $(1, 2) \neq (2, 1)$
Repeats are possible: $(1, 1) \in \mathbb{N} \times \mathbb{N}$
More Repeats

Repeating an experiment $n$ times gives the sample space

$$\Omega^n = \Omega \times \cdots \times \Omega \quad (n \text{ times})$$

$$= \{(x_1, x_2, \ldots, x_n) : x_i \in \Omega \text{ for all } i\}$$
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The element \((x_1, x_2, \ldots, x_n)\) is called an \( n \)-tuple.
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The element $$(x_1, x_2, \ldots, x_n)$$ is called an $n$-tuple.

If $|\Omega| = k$, then $|\Omega^n| = k^n$. 
Probability Rules

Complement of an event $A$

$$P(A^c) = 1 - P(A)$$

Union of two overlapping events $A \cap B \neq \emptyset$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
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Exercise

You are picking a number out of a hat, which contains the numbers 1 through 100. What are the following events and their probabilities?

- The number has a single digit
- The number has two digits
- The number is a multiple of 4
- The number is not a multiple of 4
- The sum of the number’s digits is 5
Permutations

A permutation is an ordering of an $n$-tuple. For instance, the $n$-tuple $(1, 2, 3)$ has the following permutations:

$(1, 2, 3), (1, 3, 2), (2, 1, 3)$
$(2, 3, 1), (3, 1, 2), (3, 2, 1)$
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The number of unique orderings of an \( n \)-tuple is \( n \) factorial:

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n! = n \times (n - 1) \times (n - 2) \times \cdots \times 2
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How many ways can you rearrange $(1, 2, 3, 4)$?