Notes: Independence

CS 3130/ECE 3530: Probability and Statistics for Engineers

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Independence:
An event A is independent of an event B when

$$P(A|B) = P(A)$$

In English: "the probability of A does not depend on whether B happens." If A and B are not independent, we say they are dependent.

Let's break down this equation using the definition of conditional probability:

$$\begin{aligned}
& P(A|B) = P(A) && \text{Definition of } A \text{ and } B \text{ independent} \\
\Leftrightarrow \quad & \frac{P(A \cap B)}{P(B)} = P(A) && \text{Definition of conditional prob.} \\
\Leftrightarrow \quad & P(A \cap B) = P(A)P(B) && \text{Multiply both sides by } P(B)
\end{aligned}$$

So, we see an equivalent definition of A and B being independent is that their joint probability is the product of their individual probabilities. Continuing on, we see

$$\begin{aligned}
& P(A \cap B) = P(A)P(B) && \text{Definition of } A \text{ and } B \text{ independent} \\
\Leftrightarrow \quad & \frac{P(A \cap B)}{P(A)} = P(B) && \text{Divide both sides by } P(A) \\
\Leftrightarrow \quad & P(B|A) = P(B) && \text{Definition of conditional prob.}
\end{aligned}$$

This tells us that independence is a *symmetric* property: $P(A|B) = P(A)$ is equivalent to $P(B|A) = P(B)$.

In-Class Problem: A fair die is thrown twice. A is the event sum of values is 5. And B is the event that at least one throw is a 2. Calculate $P(A \mid B)$. Are events A and B independent?

In-Class Problem: You have two urns, one with 4 black balls and 3 white balls, the other with 2 black balls and 2 white balls. You pick one urn at random and then select a ball from the urn. Is the event that I pick urn 1 independent of the event that I pick a white ball? What if I changed the second urn to have 8 black balls and 6 white balls?

In-Class Problem: You have a system with a main power supply and auxiliary power supply. The main power supply has a 10% chance of failure. If the main power supply is running, the auxiliary power supply also has a 10% chance of failure. But if the main supply fails, the auxiliary supply is more likely to be overloaded and has a 15% chance to fail. Is the auxiliary supply failing independent of main supply failing?

Looking back at our English translation of independence, we would expect (intuitively) that the probability of A would be the same if B happens or if B does *not* happen, that is, if B^c happens. Let's check if this is true:

$$
\begin{aligned}
& P(A \cap B) = P(A)P(B) && \text{Definition of } A \text{ and } B \text{ independent} \\
\Leftrightarrow\quad & P(A - B^c) = P(A)P(B) && \text{Definition of set minus} \\
\Leftrightarrow\quad & P(A) - P(A \cap B^c) = P(A)P(B) && \text{Difference rule} \\
\Leftrightarrow\quad & P(A) - P(A \cap B^c) = P(A)(1 - P(B^c)) && \text{Complement rule} \\
\Leftrightarrow\quad & P(A \cap B^c) = P(A)P(B^c) && \text{Subtract } P(A) \text{ from both sides and multiply by } -1
\end{aligned}
$$

This final line is just the definition that A and B^c are independent. To summarize, we have four different (and equivalent) definitions of independence:

Definitions of Independence

The events A and B are independent if any of the following equivalent conditions are true:

1. $P(A|B) = P(A)$
2. $P(B|A) = P(B)$
3. $P(A \cap B) = P(A)P(B)$
4. Replace B with B^c in 1-3, that is:

$$P(A|B^c) = P(A) \quad \textbf{or} \quad P(B^c|A) = P(B^c) \quad \textbf{or} \quad P(A \cap B^c) = P(A)P(B^c)$$