CS 3190: Foundations of Data Analysis **Due: Monday 9.30 at 2:45pm**

Homework 2: Convergence and Linear Algebra

Instructions: Your answers are due at 2:45, before the beginning of class on the due date. You must turn in a pdf through canvas. I recommend using latex (`http://www.cs.utah.edu/~jeffp/teaching/latex/`) for producing the assignment answers. If the answers are too hard to read you will lose points, entire questions may be given a 0 (e.g. **sloppy pictures with your phone's camera are not ok, but very careful ones are**)

Please make sure your name appears at the top of the page.

You may discuss the concepts with your classmates, but write up the answers entirely on your own. **Be sure to show all the work involved in deriving your answers! If you just give a final answer without explanation, you may not receive credit for that question.**

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1. **[30 points]** Consider a random variable X with expected values $\mathbf{E}[X] = 7$ and variance $\mathbf{Var}[X] = 2$. We would like to upper bound the probability $\mathbf{Pr}[X < 5]$.
   (a) Which bound can and cannot be used with what we know about X (Markov, Chebyshev, or Chernoff-Hoeffding), and why?
   (b) Using that bound, calculate an upper bound for $\mathbf{Pr}[X < 5]$.
   (c) Describe a probability distribution for X where the other two bounds are definitely not applicable.

2. **[30 points]** Consider n iid random variables $X_1, X_2, \ldots, X_n$ with expected value $\mathbf{E}[X_i] = 20$ and variance $\mathbf{Var}[X_i] = 2$. Assume we also know that each X_i must satisfy $15 \leq X_i \leq 22$. We now want to analyze the random variable of their average $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

   Assume first that $n = 20$ (the number of random variables).
   (a) Use the Chebyshev inequality to upper bound $\mathbf{Pr}[\bar{X} > 21]$.
   (b) Use the Chernoff-Hoeffding inequality to upper bound $\mathbf{Pr}[\bar{X} > 21]$.

   Now assume first that $n = 200$ (the number of random variables).
   (c) Use the Chebyshev inequality to upper bound $\mathbf{Pr}[\bar{X} > 21]$.
   (d) Use the Chernoff-Hoeffding inequality to upper bound $\mathbf{Pr}[\bar{X} > 21]$.

3. **[15 points]** Consider the following 2 vectors in $\mathbb{R}^4$:

$$p = (4, 2, -6, \mathtt{x})$$
$$q = (2, -4, 1, -2)$$

   Report the following:
   (a) Choose the value $\mathtt{x}$ so that p and q are orthogonal.
   (b) Calculate $\|q\|_1$
   (c) Calculate $\|q\|_2^2$

4. **[25 points]** Consider the following 2 matrices:

$$A = \begin{bmatrix} 4 & -1 & 3 \\ 0 & -1 & 4 \\ 2 & -2 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 3 & -1 \\ 5 & 3 & 5 \\ -2 & -2 & 1 \end{bmatrix}$$

Report the following:

(a) $A^T B$

(b) AB

(c) BA

(d) $B + A$

(e) Which matrices are invertable? For any that are invertable, report the result.

Practice questions

5. **[0 points]** Consider two random variables C and T describing how many coffees and teas I will buy in the coming week; clearly neither can be smaller than 0. Based on personal experience, I know the following summary statistics about my coffee and tea buying habits: $\textsf{\textbf{E}}[C] = 3$ and $\textsf{\textbf{Var}}[C] = 1$ also $\textsf{\textbf{E}}[T] = 2$ and $\textsf{\textbf{Var}}[T] = 5$.

(a) Use Markov's Inequality to upper bound the probability I buy 4 or more coffees, and the same for teas: $\textsf{\textbf{Pr}}[C \geq 4]$ and $\textsf{\textbf{Pr}}[T \geq 4]$.

(b) Use Chebyshev's Inequality to upper bound the probability I buy 4 or more coffees, and the same for teas: $\textsf{\textbf{Pr}}[C \geq 4]$ and $\textsf{\textbf{Pr}}[T \geq 4]$.

6. **[0 points]** Consider a parked self-driving car that returns n iid estimates to the distance of a tree. We will model these n estimates as a set of n scalar random variables $X_1, X_2, \ldots, X_n$ taken iid from an unknown pdf f, which we assume models the true distance plus unbiased noise. (The sensor can take many iid estimates in rapid fire fashion.) The sensor is programmed to only return values between 0 and 20 feet, and that the variance of the sensing noise is 64 feet squared. Let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. We want to understand as a function of n how close $\bar{X}$ is to μ, which is the true distance to the car.

(a) Use Chebyshev's Inequality to determine a value n so that $\textsf{\textbf{Pr}}[|\bar{X} - \mu| \geq 1] \leq 0.5$.

(b) Use Chebyshev's Inequality to determine a value n so that $\textsf{\textbf{Pr}}[|\bar{X} - \mu| \geq 0.1] \leq 0.1$.

(c) Use the Chernoff-Hoeffding bound to determine a value n so that $\textsf{\textbf{Pr}}[|\bar{X} - \mu| \geq 1] \leq 0.5$.

(d) Use the Chernoff-Hoeffding bound to determine a value n so that $\textsf{\textbf{Pr}}[|\bar{X} - \mu| \geq 0.1] \leq 0.1$.

7. **[0 points]** Let X be a random variable that you know is in the range $[-1, 2]$ and you know has expected value of $\textsf{\textbf{E}}[X] = 0$. Use the Markov Inequality to upper bound $\textsf{\textbf{Pr}}[X > 1.5]$? *(Hint: you will need to use a change of variables.)*

8. **[0 points]** Consider a matrix

$$A = \begin{bmatrix} 2 & 2 & 3 \\ -2 & 7 & 4 \\ -3 & -3 & -6 \\ -8 & 2 & 3 \end{bmatrix}.$$

(a) Add a column to A so that it is invertable.

(b) Remove a row from A so that it is invertable.

(c) Is AA^T invertable?

(d) Is A^TA invertable?