In the last lecture we looked at the smoothing properties of the Jacobi smoother

\[ \lambda_k^{\text{new}} = \lambda_k^{\text{old}} \cos \left( \frac{k\pi}{N} \right) \]

Damped Jacobi Smoother

Recall,

\[ e_i^{\text{new}} = \frac{1}{2} (e_i^{\text{old}} + e_{i+1}^{\text{old}}) \]

\[ = \lambda_k^{\text{old}} \cos \frac{k\pi}{N} \sin \frac{k\pi i}{N} \]

Now, let us incorporate a damping term \( w \)

\[ e_i^{\text{new}} = e_i^{\text{old}} + w (e_i^{\text{new}} - e_i^{\text{old}}) \]

\[ e_i^{\text{new}} = \lambda_k^0 \sin \frac{k\pi i}{N} + w (\lambda_k^0 (\cos \frac{k\pi}{N} - 1) \sin \frac{k\pi i}{N}) \]

\[ = \lambda_k^0 (1 + w \cos \frac{k\pi}{N} - w) \sin \frac{k\pi i}{N} \]

\[ \lambda_k^{\text{new}} = \lambda_k^0 (1 - w + w \cos \frac{k\pi}{N}) \]
\[e^m = R^m e_0\]

Using coarse grids to accelerate the computation
Nested Iteration

1. Use coarse to generate initial guess for fine
   - faster
   - work / iter
   - # iteration \((k \rightarrow 2k)\)

2. Use coarse grid to eliminate smooth on low-freq components of error.

Project eigen vector into wave number \(k = 4\)
on \(S_n \rightarrow \mathbb{R}^{2n}\)

\[ V_k^n = \sin (j k \pi h) \]

\[ k = 4 \leq 6 = \frac{12}{2} \]

Smooth \((\frac{N}{2})\)

\[ V_k^n (2k) = \sin (2j k \pi h) = \sin (j 2k \pi (2h)) \]

Rough component:

\[ k = 4 \geq 6 \leq 3 \]

* Projection to “coarse grid” transforms our smooth error components to oscillatory ones.

\[ \Rightarrow \text{Combined with iterative smoother} \]

Aliasing

Eigen vectors with wavelength \(< 2h\) upper smooth, other wise using are advised
\[ K = N \]

\[ V_n = \sin \left( iN \pi \frac{n}{N} \right) = \sin \left( i \pi \frac{n}{N} \right) = 0 \]

\[ 1 < m < N \]

\[ V_{m+N} = \sin \left( i(N+m)\pi \frac{1}{N} \right) = \sin \left( i \left( 1 + \frac{m}{N} \right) \pi \right) \]

\[ = \sin (i\pi) \cos \left( i \frac{m}{N} \pi \right) + \cos (i\pi) \sin \left( i \frac{m}{N} \pi \right) \]

\[ V_{N-m} = -V_{N+m} \quad \text{ALIASING} \]

\[ E_n = A^{-1} r_n \]
MULTIGRID

- Smooth \( A u = f \) on \( S^h \) to compute \( v^h \)
- Compute the residual \( r^h = f^h - A^h v^h \)
- \( \text{RESTRICT} \) \( r^{2h} \leftarrow r^h \)
- Smooth \( A e^{2h} = r^{2h} \) on \( S^{2h} \)
- \( \text{PROLONG} \) \( e^h \leftarrow e^{2h} \)
- Correct \( v^h = v^h + e^h \)
- Smooth \( A u = f \) on \( S^h \) using \( v^h \)

Restriction, Prolongation
\( S^h \rightarrow S^{2h} \), \( S^{2h} \rightarrow S^h \)

Mapping coarse to fine