Map Reduce
Last Time …

- Parallel Algorithms
  - Work/Depth Model
- Spark
- Map Reduce
- Assignment 1

- Questions ?
Today ...

- Map Reduce
  - Matrix multiplication
  - Similarity Join
  - Complexity theory
MapReduce – word counting

- **Input**: set of documents
- **Map**:
  - reads a document and breaks it into a sequence of words \( w_1, w_2, ..., w_n \)
  - Generates \((k, v)\) pairs, \((w_1, 1), (w_2, 1), ..., (w_n, 1)\)
- **System**:  
  - group all \((k, v)\) by key  
  - Given \(r\) reduce tasks, assign keys to reduce tasks using a hash function
- **Reduce**:  
  - Combine the values associated with a given key  
  - Add up all the values associated with the word \(\rightarrow\) total count for that word
Matrix-vector multiplication

- \( n \times n \) matrix \( M \) with entries \( m_{ij} \)
- Vector \( v \) of length \( n \) with values \( v_j \)
- We wish to compute
  \[
  x_i = \sum_{j=1}^{n} m_{ij} v_j
  \]
  - If \( v \) can fit in memory
    - Map: generate \((i, m_{ij} v_j)\)
    - Reduce: sum all values of \( i \) to produce \((i, x_i)\)
  - If \( v \) is too large to fit in memory? Stripes? Blocks?
  - What if we need to do this iteratively?
Matrix-Matrix Multiplication

- $P = MN \rightarrow p_{ik} = \sum_j m_{ij} n_{jk}$
- 2 mapreduce operations
  - Map 1: produce $(k, v), \left(j, (M, i, m_{ij})\right)$ and $(j, (N, k, n_{jk}))$
  - Reduce 1: for each $j \rightarrow (i, k, m_{ij} \times n_{jk})$
  - Map 2: identity
  - Reduce 2: sum all values associated with key $(i, k)$
Matrix-Matrix multiplication

- In one mapreduce step
  - Map:
    - generate \((k, v) \rightarrow ((i, k), (M, j, m_{ij})) \& ((i, k), (N, j, n_{jk}))\)
  - Reduce:
    - each key \((i, k)\) will have values \(((i, k), (M, j, m_{ij})) \& ((i, k), (N, j, n_{jk}))\) \(\forall j\)
    - Sort all values by \(j\)
    - Extract \(m_{ij}\) & \(n_{jk}\) and multiply, accumulate the sum
Complexity Theory for mapreduce
Communication cost

- Communication cost of a task is the size of the input to the task.
- We do not consider the amount of time it takes each task to execute when estimating the running time of an algorithm.
- The algorithm output is rarely large compared with the input or the intermediate data produced by the algorithm.
Reducer size & Replication rate

- **Reducer size** ($q$)
  - Upper bound on the number of values that are allowed to appear in the list associated with a single key
  - By making the reducer size small, we can force there to be many reducers
    - High parallelism $\rightarrow$ low wall-clock time
  - By choosing a small $q$ we can perform the computation associated with a single reducer entirely in the main memory of the compute node
    - Low synchronization (Comm/IO) $\rightarrow$ low wall clock time

- **Replication rate** ($r$)
  - number of $(k, v)$ pairs produced by all the Map tasks on all the inputs, divided by the number of inputs
  - $r$ is the average communication from Map tasks to Reduce tasks
Example: one-pass matrix mult

- Assume matrices are $n \times n$
- $r$ – replication rate
  - Each element $m_{ij}$ produces $n$ keys
  - Similarly each $n_{jk}$ produces $n$ keys
  - Each input produces exactly $n$ keys $\rightarrow$ load balance
- $q$ – reducer size
  - Each key has $n$ values from $M$ and $n$ values from $N$
  - $2n$
Example: two-pass matrix mult

- Assume matrices are $n \times n$
- $r$ – replication rate
  - Each element $m_{ij}$ produces 1 key
  - Similarly each $n_{jk}$ produces 1 key
  - Each input produces exactly 1 key (2nd pass)
- $q$ – reducer size
  - Each key has $n$ values from $M$ and $n$ values from $N$
  - $2n$ (1st pass), $n$ (2nd pass)
Real world example: Similarity Joins

- Given a large set of elements $X$ and a similarity measure $s(x, y)$
- Output: pairs whose similarity exceeds a given threshold $t$
- Example: given a database of $10^6$ images of size 1MB each, find pairs of images that are similar
- Input: $(i, P_i)$, where $i$ is an ID for the picture and $P_i$ is the image
- Output: $(P_i, P_j)$ or simply $(i, j)$ for those pairs where $s(P_i, P_j) > t$
Approach 1

- Map: generate \((k, v)\)

\[
(\langle i, j \rangle, \langle P_i, P_j \rangle)
\]

- Reduce:
  - Apply similarity function to each value (image pair)
  - Output pair if similarity above threshold \(t\)

- Reducer size – \(q\) → 2 (2MB)
- Replication rate – \(r\) → \(10^6 - 1\)
- Total communication from map→reduce tasks?
  - \(10^6 \times 10^6 \times 10^6\) bytes → \(10^{18}\) bytes → 1 Exabyte (kB MB GB TB PB EB)
  - Communicate over GigE → \(10^{10}\) sec → 300 years
Approach 2: group images

- Group images into $g$ groups with $\frac{10^6}{g}$ images each
- Map: Take input element $(i, P_i)$ and generate
  - $(g - 1)$ keys $(u, v) | P_i \in \mathcal{G}(u), \ v \in \{1, ..., g\} \setminus \{u\}$
  - Associated value is $(i, P_i)$
- Reduce: consider key $(u, v)$
  - Associated list will have $2 \times \frac{10^6}{g}$ elements $(j, P_j)$
  - Take each $(i, P_i)$ and $(j, P_j)$ where $i, j$ belong to different groups and compute $s(P_i, P_j)$
  - Compare pictures belonging to the same group
    - heuristic for who does this, say reducer for key $(u, u + 1)$
Approach 2: group images

- Replication rate: \( r = g - 1 \)
- Reducer size: \( q = 2 \times 10^6 / g \)
- Input size: \( 2 \times 10^{12} / g \) bytes

Say \( g = 1000 \),
  - Input is 2GB
  - Total communication: \( 10^6 \times 999 \times 10^6 = 10^{15} \) bytes \( \rightarrow \) 1 petabyte
Graph model for mapreduce problems

- Set of inputs
- Set of outputs
- many-many relationship between the inputs and outputs, which describes which inputs are necessary to produce which outputs.

- Mapping schema
  - Given a reducer size $q$
  - No reducer is assigned more than $q$ inputs
  - For every output, there is at least one reducer that is assigned all inputs related to that output
Grouping for Similarity Joins

- Generalize the problem to \( p \) images
- \( g \) equal sized groups of \( \frac{p}{g} \) images
- Number of outputs is \( \binom{p}{2} \approx \frac{p^2}{2} \)
- Each reducer receives \( \frac{2p}{g} \) inputs \((q)\)
- Replication rate \( r = g - 1 \)

\[ r = \frac{2p}{q} \]

- The smaller the reducer size, the larger the replication rate, and therefore higher the communication
  - communication ↔ reducer size
  - communication ↔ parallelism
Lower bounds on Replication rate

1. Prove an upper bound on how many outputs a reducer with \( q \) inputs can cover. Call this bound \( g(q) \)

2. Determine the total number of outputs produced by the problem

3. Suppose that there are \( k \) reducers, and the \( i^{th} \) reducer has \( q_i < q \) inputs. Observe that \( \sum_{i=1}^{k} g(q_i) \) must be no less than the number of outputs computed in step 2

4. Manipulate inequality in 3 to get a lower bound on \( \sum_{i=1}^{k} q_i \)

5. 4 is the total communication from Map tasks to reduce tasks. Divide by number of inputs to get the replication rate
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\[
q \geq \frac{p}{q}
\]
Matrix Multiplication

- Consider the one-pass algorithm → extreme case
- Lets group rows/columns into bands → $g$ groups → $n/g$ columns/rows
Matrix Multiplication

- **Map:**
  - for each element of $M, N$ generate $g$ $(k, v)$ pairs
  - Key is group paired with all groups
  - Value is $(i, j, m_{ij})$ or $(i, j, n_{ij})$

- **Reduce:**
  - Reducer corresponds to key $(i, j)$
  - All the elements in the $i^{th}$ band of $M$ and $j^{th}$ band of $N$
  - Each reducer gets $n \left(\frac{n}{g}\right)$ elements from 2 matrices

\[ q = \frac{2n^2}{g}, \quad r = g \Rightarrow r = \frac{2n^2}{q} \]
Lower bounds on Replication rate

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2. Determine the total number of outputs produced by the problem

3. Suppose that there are $k$ reducers, and the $i^{th}$ reducer has $q_i < q$ inputs. Observe that $\sum_{i=1}^{k} g(q_i)$ must be no less than the number of outputs computed in step 2

4. Manipulate inequality in 3 to get a lower bound on $\sum_{i=1}^{k} q_i$

5. 4 is the total communication from Map tasks to reduce tasks. Divide by number of inputs to get the replication rate

$\Rightarrow$ Each reducer receives $k$ rows from $M$ and $N \rightarrow q = 2nk$ and produces $k^2$ outputs $\Rightarrow g(q) = \frac{q^2}{4n^2}$

$\Rightarrow n^2$

$\Rightarrow \sum_{i=1}^{k} q_i^2 \geq n^2$

$\Rightarrow \sum_{i=1}^{k} q_i^2 \geq 4n^4$

$\Rightarrow \sum_{i=1}^{k} q_i \geq \frac{4n^3}{q}$

$\Rightarrow r = \frac{1}{2n^2} \sum_{i=1}^{k} q_i = \frac{2n^2}{q}$
Matrix Multiplication

LET US REVISIT THE TWO-PASS APPROACH
Matrix-Matrix Multiplication

- \( P = MN \rightarrow p_{ik} = \sum_j m_{ij}n_{jk} \)

- 2 mapreduce operations
  - Map 1: produce \((k, v), (j, (M, i, m_{ij}))\) and \((j, (N, k, n_{jk}))\)
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Grouped two-pass approach

\[ g^2 \text{ groups of } \frac{n^2}{g^2} \text{ elements each} \]

First pass: compute products of square \((I, J)\) of \(M\) with square \((J, K)\) of \(N\)

Second pass: \(\forall I, K\) sum over all \(J\)
Grouped two-pass approach

- Replication rate for map1 is $g \rightarrow 2gn^2$ total communication
- Each reducer gets $\frac{2n^2}{g^2} \rightarrow q = \frac{2n^2}{g^2} \rightarrow g = n \sqrt{\frac{2}{q}}$
- Total communication $\rightarrow 2 \frac{\sqrt{2}n^3}{\sqrt{q}}$
- Assume map2 runs on same nodes as reduce1 $\rightarrow$ no communication
- Communication $\rightarrow gn^2 \rightarrow \frac{\sqrt{2}n^3}{\sqrt{q}}$
- Total communication $\rightarrow 3 \frac{\sqrt{2}n^3}{\sqrt{q}}$
Comparison

\[
\frac{n^4}{q} < \frac{n^3}{\sqrt{q}}
\]

If \( q \) is closer to the minimum of \( 2n \), two pass is better by a factor of \( \Theta(\sqrt{n}) \).