

# Hidden Markov Models

Many slides courtesy of  
Dan Klein, Stuart Russell,  
or Andrew Moore

**CS 5300 / CS 6300**  
**Artificial Intelligence**  
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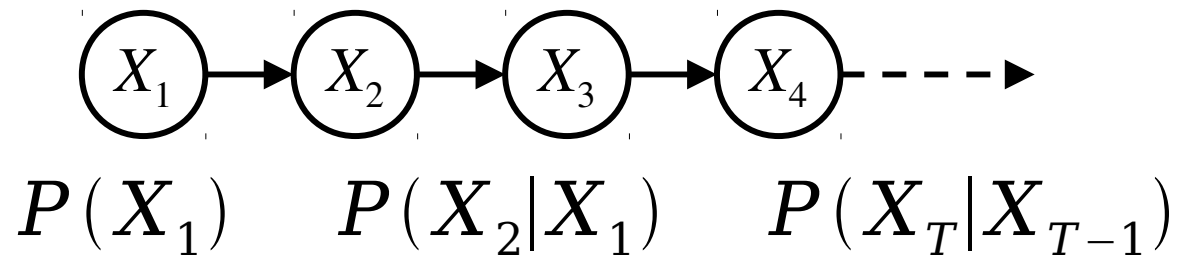
[www.cs.utah.edu/~hal/courses/2010S\\_AI](http://www.cs.utah.edu/~hal/courses/2010S_AI)

# Reasoning over Time

- Often, we want to **reason about a sequence** of observations
  - Speech recognition
  - Robot localization
  - User attention
  - Medical monitoring
- Need to introduce time into our models
- Basic approach: hidden Markov models (HMMs)
- More general: dynamic Bayes' nets

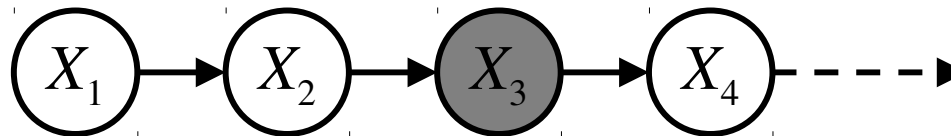
# Markov Models

- A **Markov model** is a chain-structured BN
  - Each node is identically distributed (stationarity)
  - Value of  $X$  at a given time is called the **state**
  - As a BN:



- Parameters: called **transition probabilities** or dynamics, specify how the state evolves over time (also, initial probs)

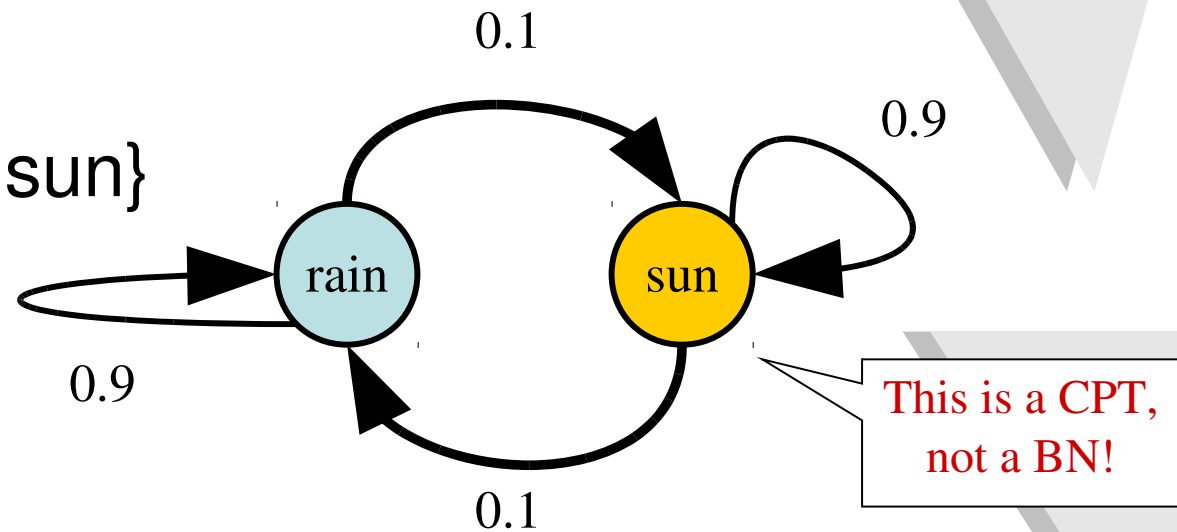
# Conditional Independence



- Basic conditional independence:
  - Past and future independent of the present
  - Each time step only depends on the previous
  - This is called the (first order) Markov property
- Note that the chain is just a (growing) BN
  - We can always use generic BN reasoning on it (if we truncate the chain)

# Example: Markov Chain

- Weather:
- States:  $X = \{\text{rain}, \text{sun}\}$
- Transitions:



- Initial distribution: 1.0 sun
- What's the probability distribution after one step?

$$\begin{aligned}
 P(X_2 = \text{sun}) &= P(X_2 = \text{sun} | X_1 = \text{sun})P(X_1 = \text{sun}) + \\
 &P(X_2 = \text{sun} | X_1 = \text{rain})P(X_1 = \text{rain}) \\
 &0.9 \cdot 1.0 + 0.1 \cdot 0.0 = 0.9
 \end{aligned}$$

# Mini-Forward Algorithm

- Question: probability of being in state  $x$  at time  $t$ ?
- Slow answer:
  - Enumerate all sequences of length  $t$  which end in  $s$
  - Add up their probabilities

$$P(X_t = sun) = \sum_{x_1 \dots x_{t-1}} P(x_1, \dots, x_{t-1}, sun)$$

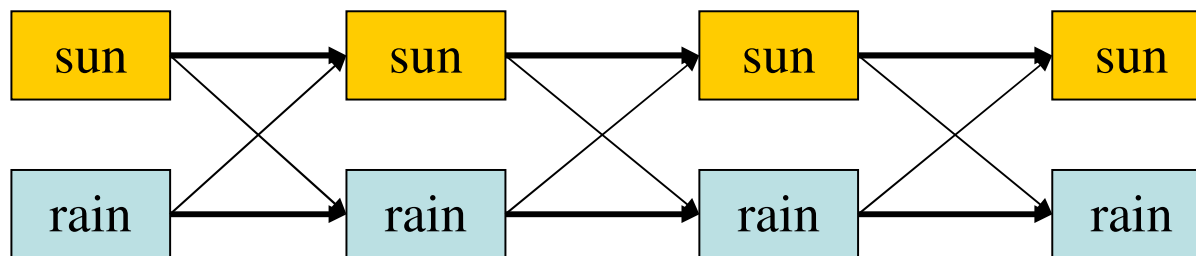
$$P(X_1 = sun)P(X_2 = sun|X_1 = sun)P(X_3 = sun|X_2 = sun)P(X_4 = sun|X_3 = sun)$$

$$P(X_1 = sun)P(X_2 = rain|X_1 = sun)P(X_3 = sun|X_2 = rain)P(X_4 = sun|X_3 = sun)$$

⋮

# Mini-Forward Algorithm

- Better way: cached incremental belief updates
  - An instance of variable elimination!



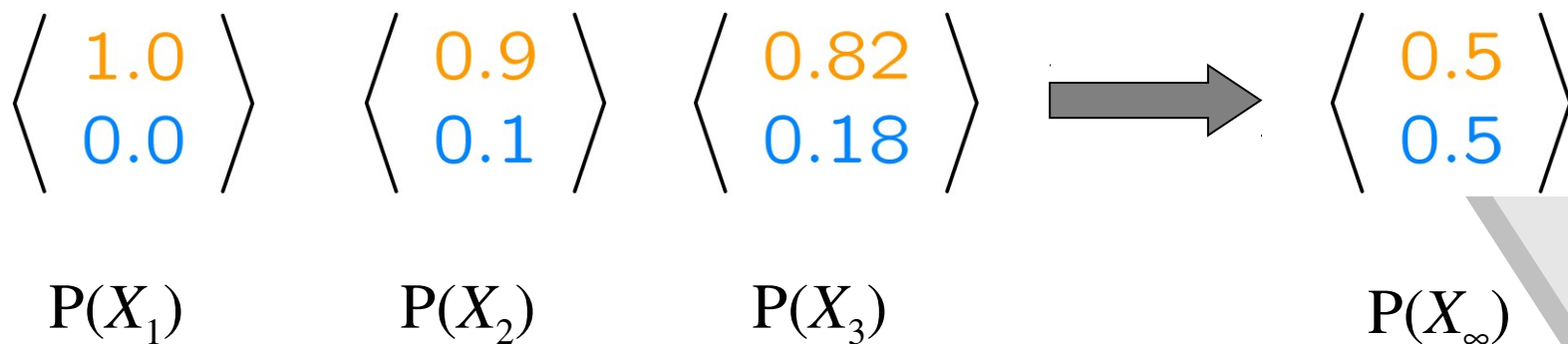
$$P(x_1) = \text{known}$$

$$P(x_t) = \sum_{x_{t-1}} P(x_t | x_{t-1}) P(x_{t-1})$$

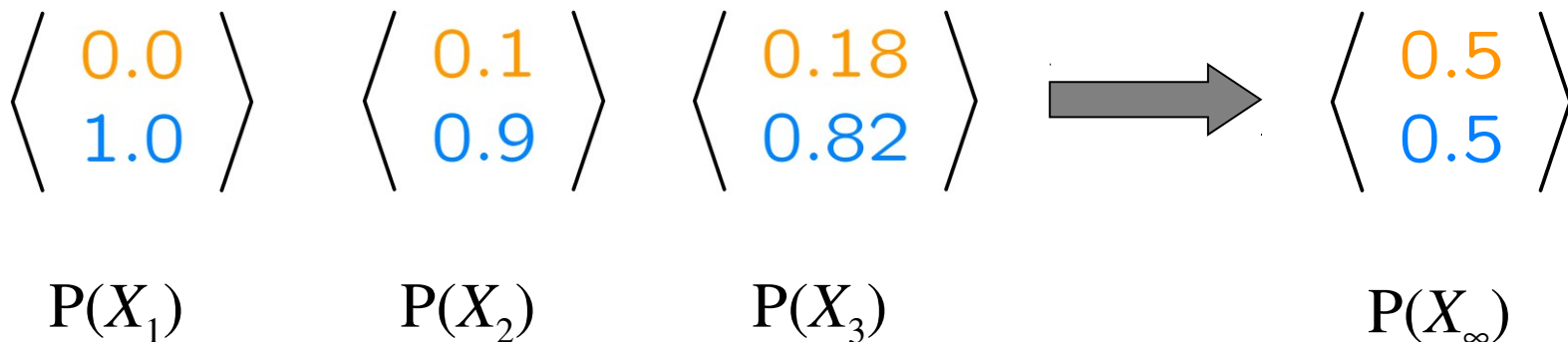
← *Forward simulation*

# Example

- From initial observation of sun



- From initial observation of rain

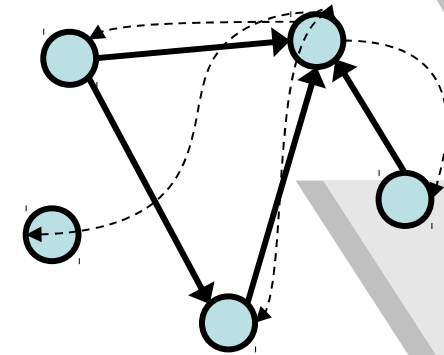


# Stationary Distributions

- If we simulate the chain long enough:
  - What happens?
  - Uncertainty accumulates
  - Eventually, we have no idea what the state is!
  
- Stationary distributions:
  - For most chains, the distribution we end up in is independent of the initial distribution (but not always uniform!)
  - Called the **stationary distribution** of the chain
  - Usually, can only predict a short time out

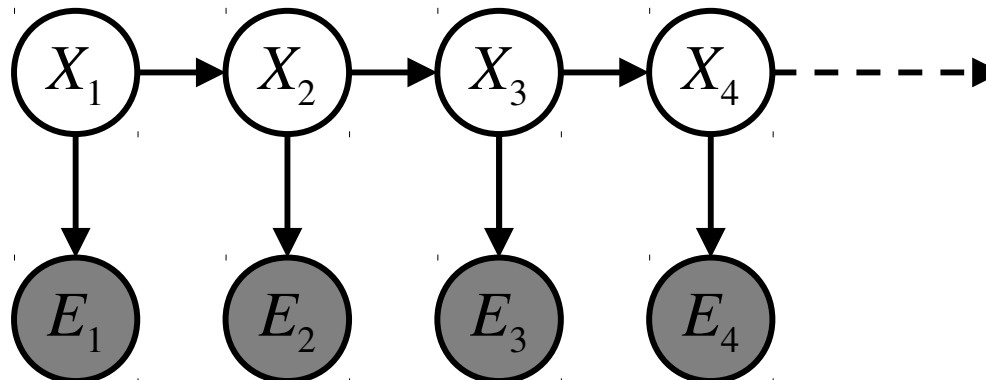
# Web Link Analysis

- PageRank over a web graph
  - Each web page is a state
  - Initial distribution: uniform over pages
  - Transitions:
    - With prob.  $c$ , uniform jump to a random page (dotted lines)
    - With prob.  $1-c$ , follow a random outlink (solid lines)
- Stationary distribution
  - Will spend more time on highly reachable pages
  - E.g. many ways to get to the Acrobat Reader download page
  - Somewhat robust to link spam (but not immune)
  - Google 1.0 returned the set of pages containing all your keywords in decreasing rank, now all search engines use link analysis along with many other factors

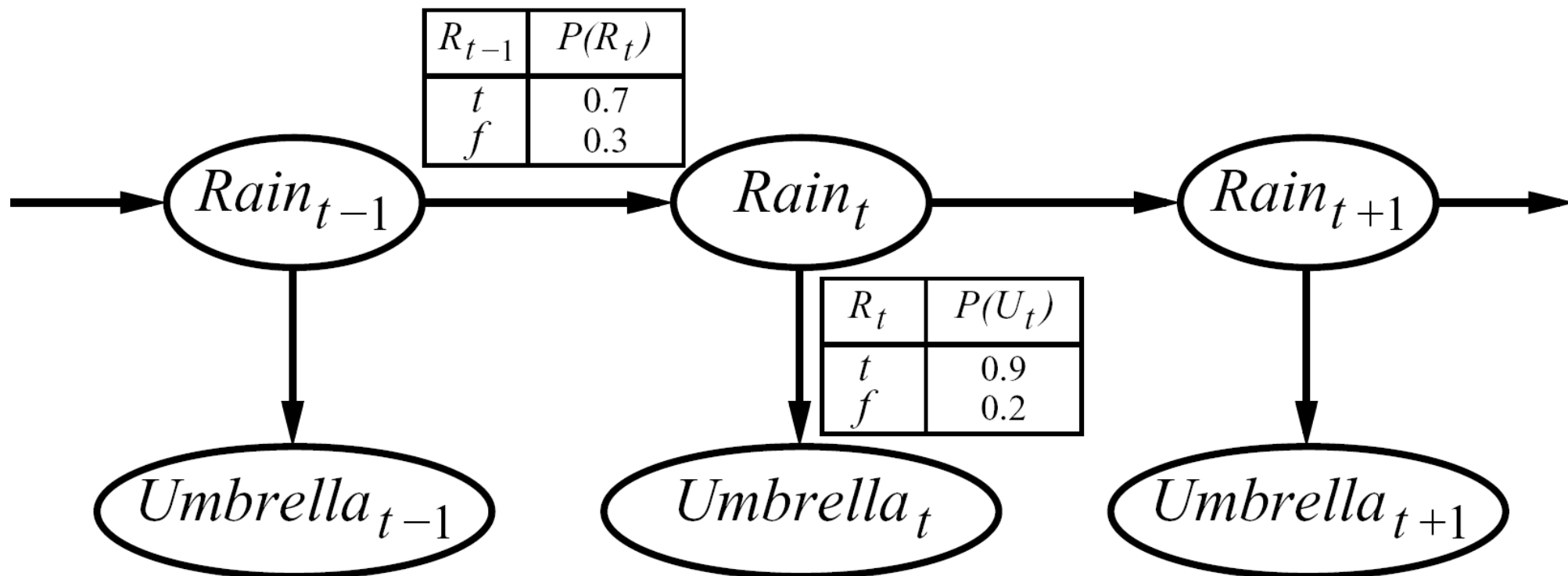


# Hidden Markov Models

- Markov chains not so useful for most agents
  - Eventually you don't know anything anymore
  - Need observations to update your beliefs
  
- Hidden Markov models (HMMs)
  - Underlying Markov chain over states  $S$
  - You observe outputs (effects) at each time step
  - As a Bayes' net:



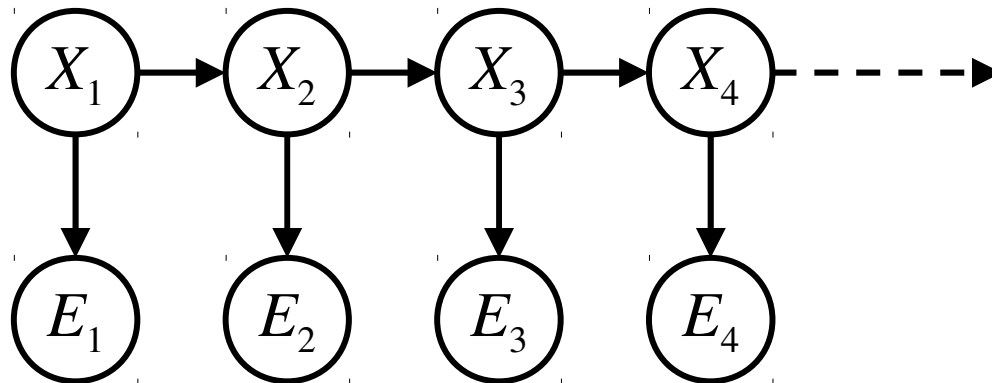
# Example



- An HMM is defined by:
  - Initial distribution:  $P(X_1)$
  - Transitions:  $P(X_T | X_{T-1})$
  - Emissions:  $P(E | X)$

# Conditional Independence

- HMMs have two important independence properties:
  - Markov hidden process, future depends on past via the present
  - Current observation independent of all else given current state



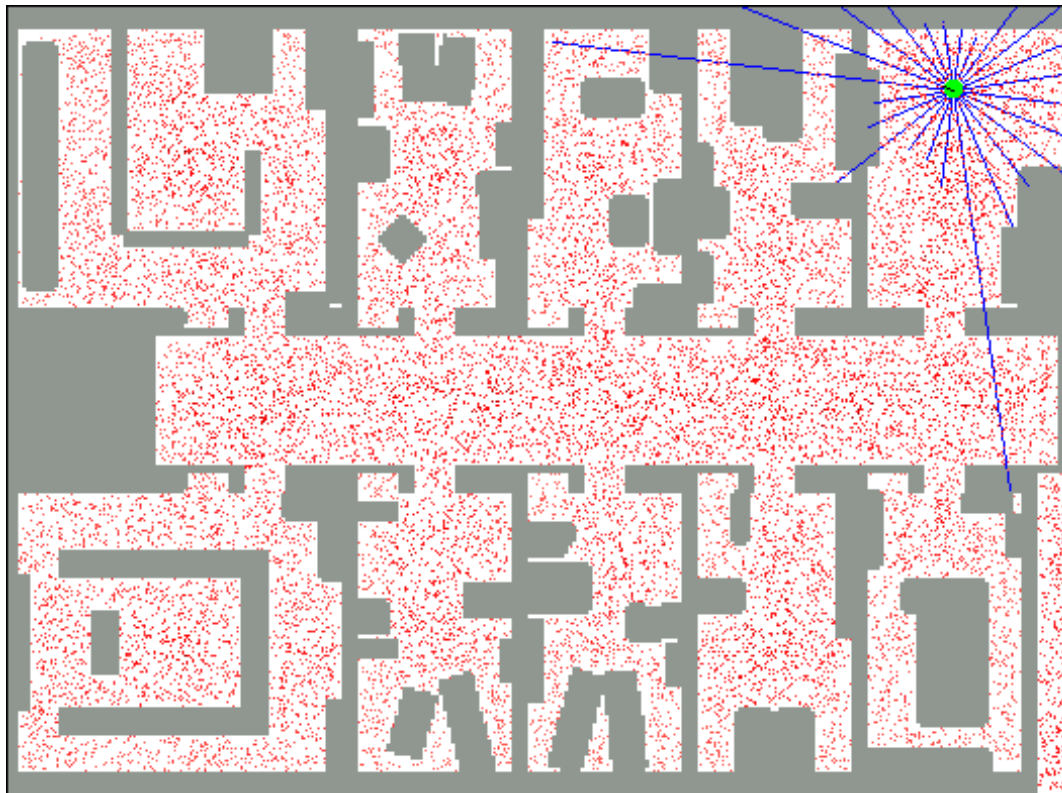
- Quiz: does this mean that observations are independent given no evidence?
  - [No, correlated by the hidden state]

# Real HMM Examples

- Speech recognition HMMs:
  - Observations are acoustic signals (continuous valued)
  - States are specific positions in specific words (so, tens of thousands)
  
- Machine translation HMMs:
  - Observations are words (tens of thousands)
  - States are translation options
  
- Robot tracking:
  - Observations are range readings (continuous)
  - States are positions on a map (continuous)

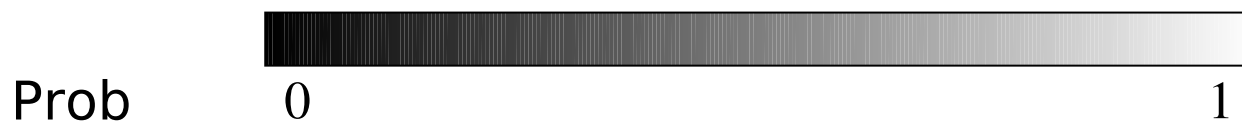
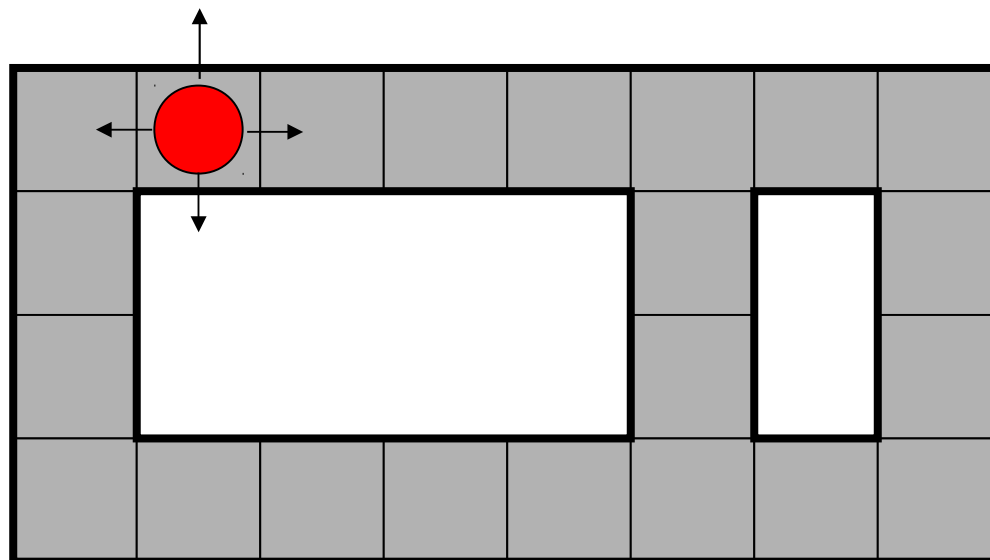
# Filtering / Monitoring

- Filtering, or monitoring, is the task of tracking the distribution  $B(X)$  (the belief state)
- We start with  $B(X)$  in an initial setting, usually uniform
- As time passes, or we get observations, we update  $B(X)$



# Example: Robot Localization

*Example from Michael Pfeiffer*

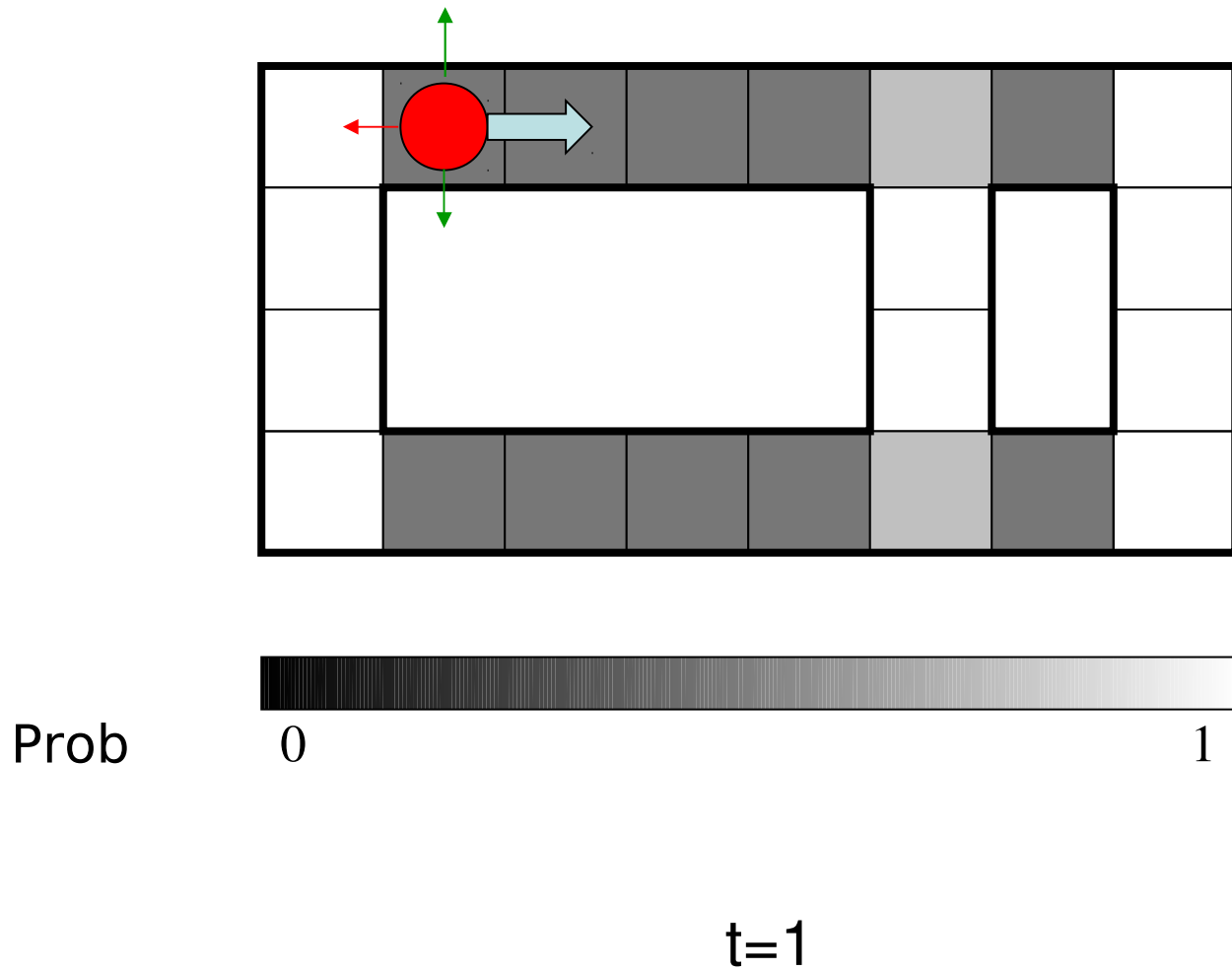


$t=0$

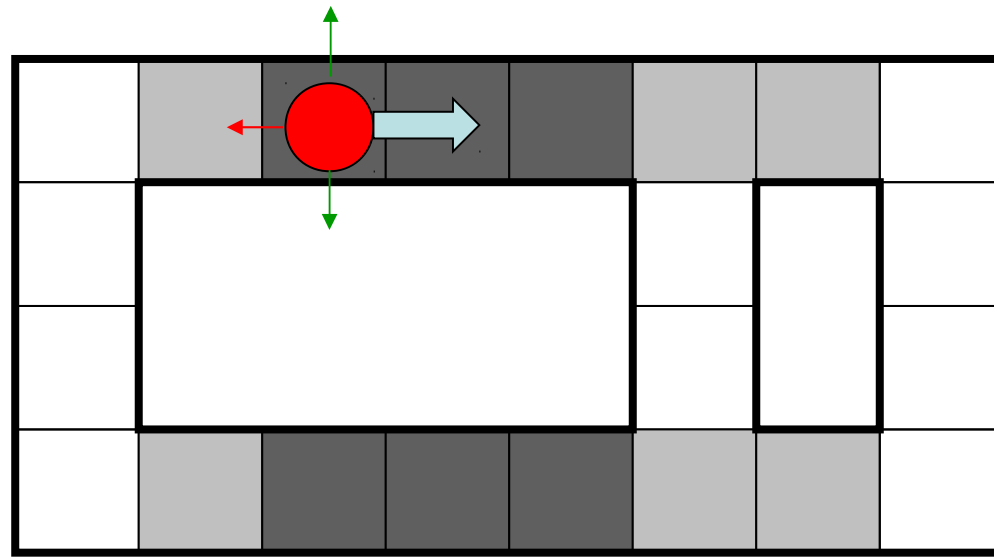
Sensor model: never more than 1 mistake

Motion model: may not execute action with small prob.

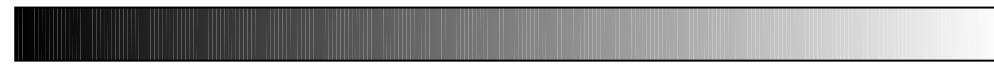
# Example: Robot Localization



# Example: Robot Localization



Prob

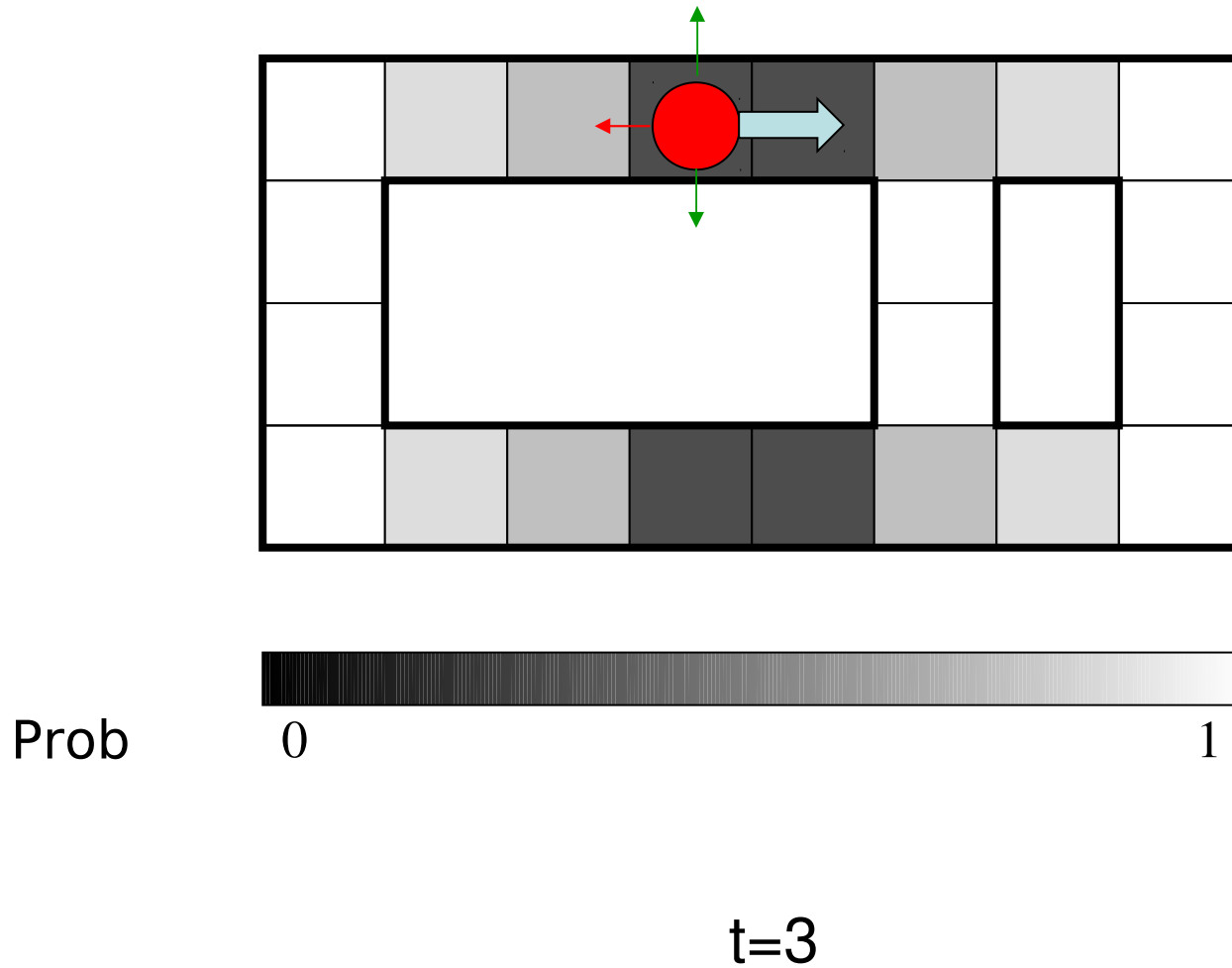


0

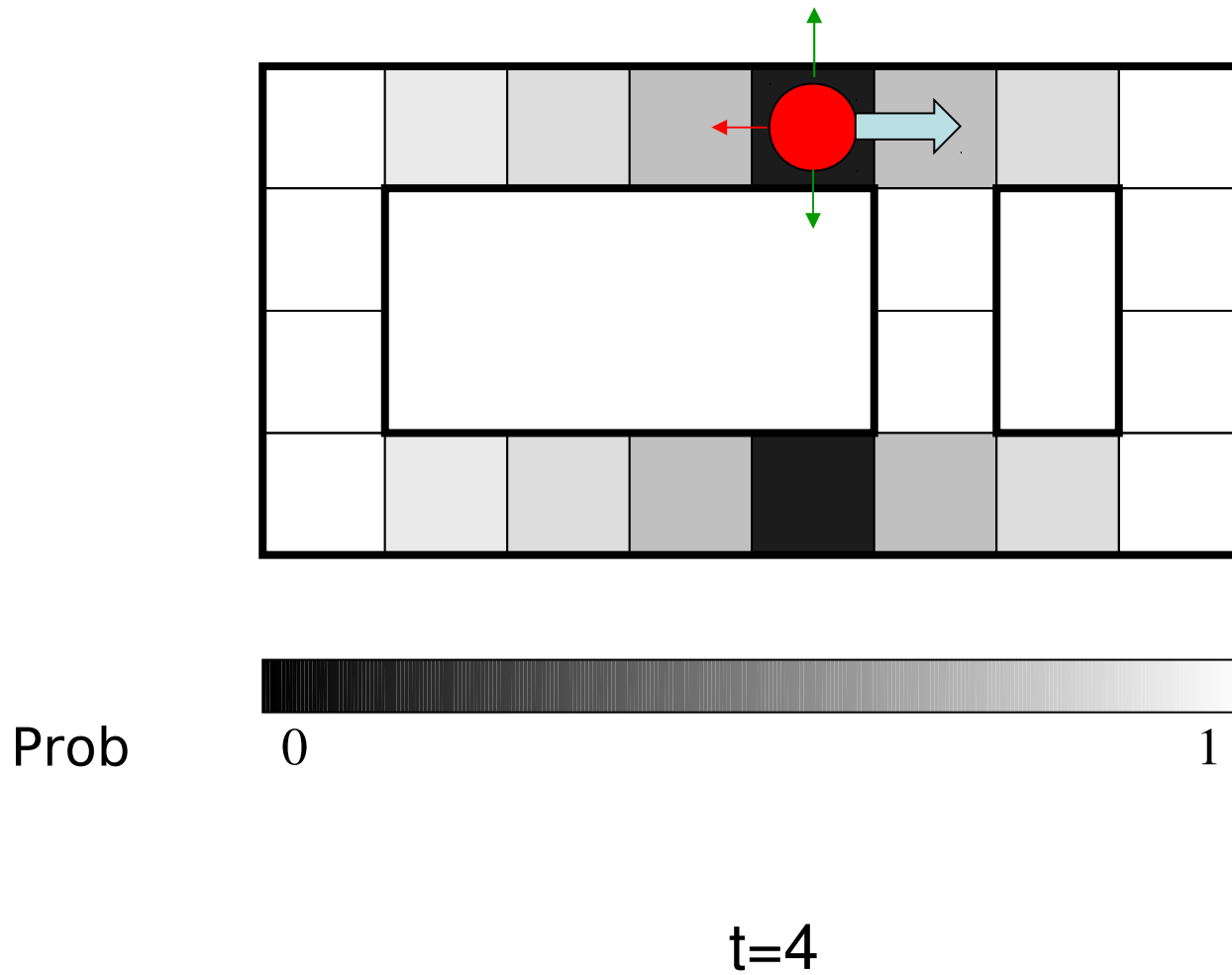
1

t=2

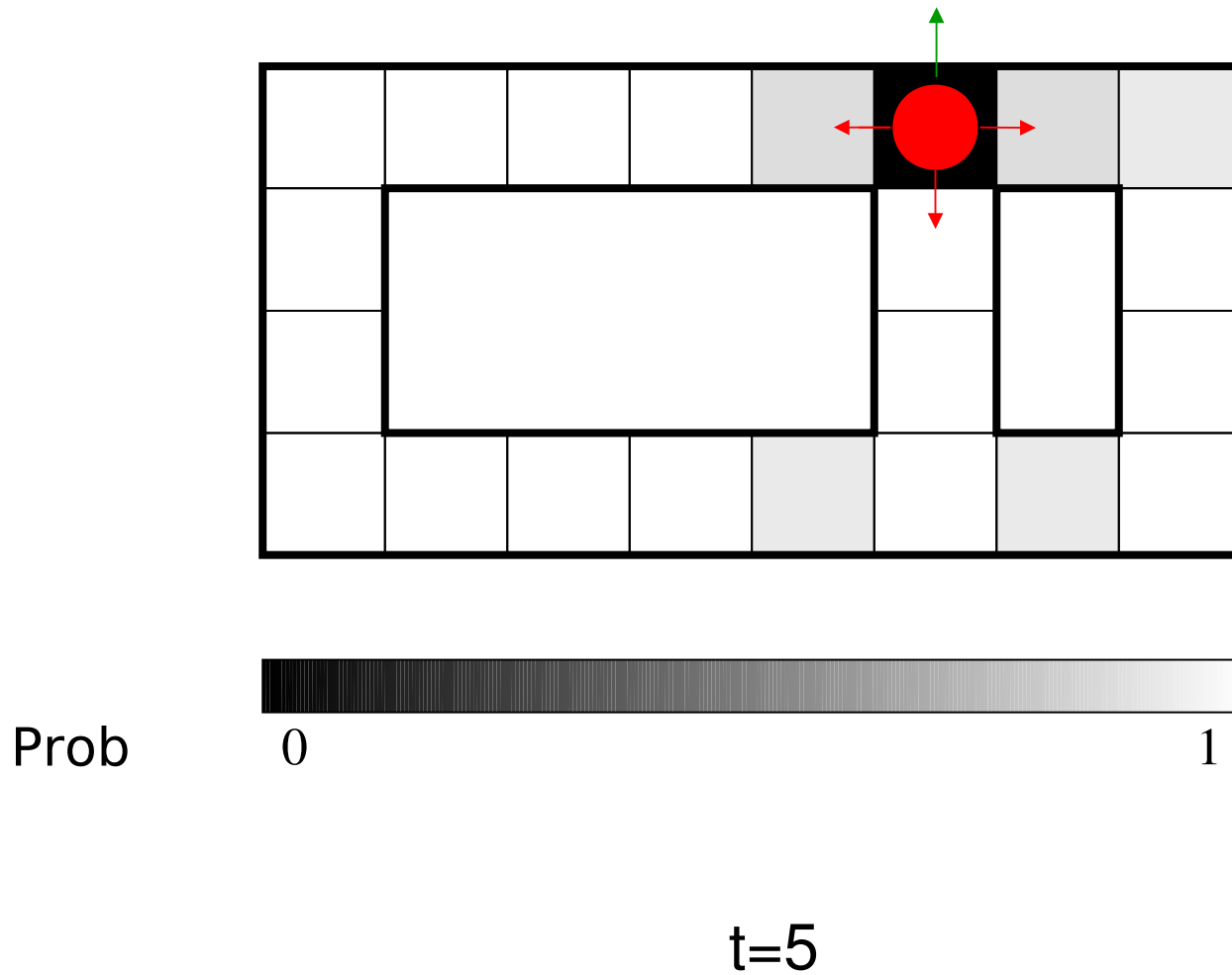
# Example: Robot Localization



# Example: Robot Localization



# Example: Robot Localization



# Passage of Time

- Assume we have current belief  $P(X \mid \text{evidence to date})$

$$B(X_t) = P(X_t | e_{1:t})$$

- Then, after one time step passes:

$$P(X_{t+1} | e_{1:t}) = \sum_{x_t} P(X_{t+1} | x_t) P(x_t | e_{1:t})$$

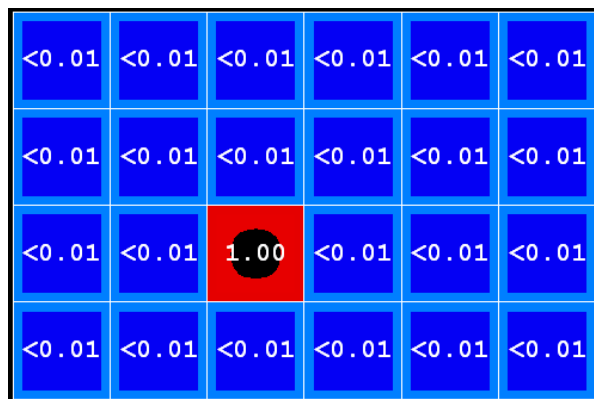
- Or, compactly:

$$B'(X_{t+1}) = \sum_{x_t} P(X' | x) B(x_t)$$

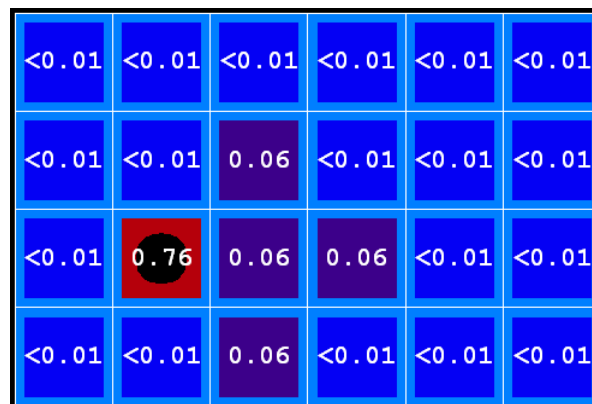
- Basic idea: beliefs get “pushed” through the transitions
  - With the “B” notation, we have to be careful about what time step  $t$  the belief is about, and what evidence it includes

# Example: Passage of Time

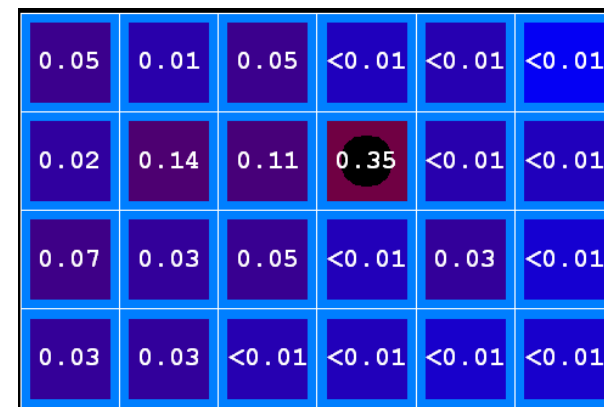
- As time passes, uncertainty “accumulates”



T = 1



T = 2



T = 5

$$B'(X) = \sum_x P(X'|x)B(x)$$

Transition model: ships usually go clockwise

# Observation

- Assume we have current belief  $P(X \mid \text{previous evidence})$ :

$$B'(X_{t+1}) = P(X_{t+1} | e_{1:t})$$

- Then:

$$P(X_{t+1} | e_{1:t+1}) \propto P(e_{t+1} | X_{t+1}) P(X_{t+1} | e_{1:t})$$

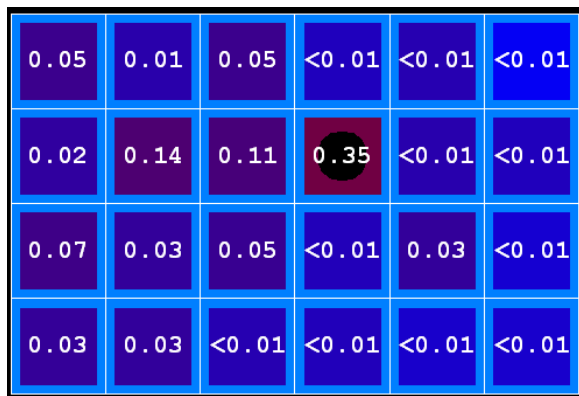
- Or:

$$B(X_{t+1}) \propto P(e | X) B'(X_{t+1})$$

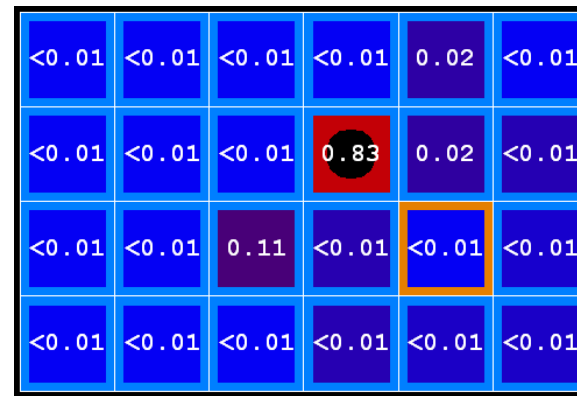
- Basic idea: beliefs reweighted by likelihood of evidence
- Unlike passage of time, we have to renormalize

# Example: Observation

- As we get observations, beliefs get reweighted, uncertainty “decreases”



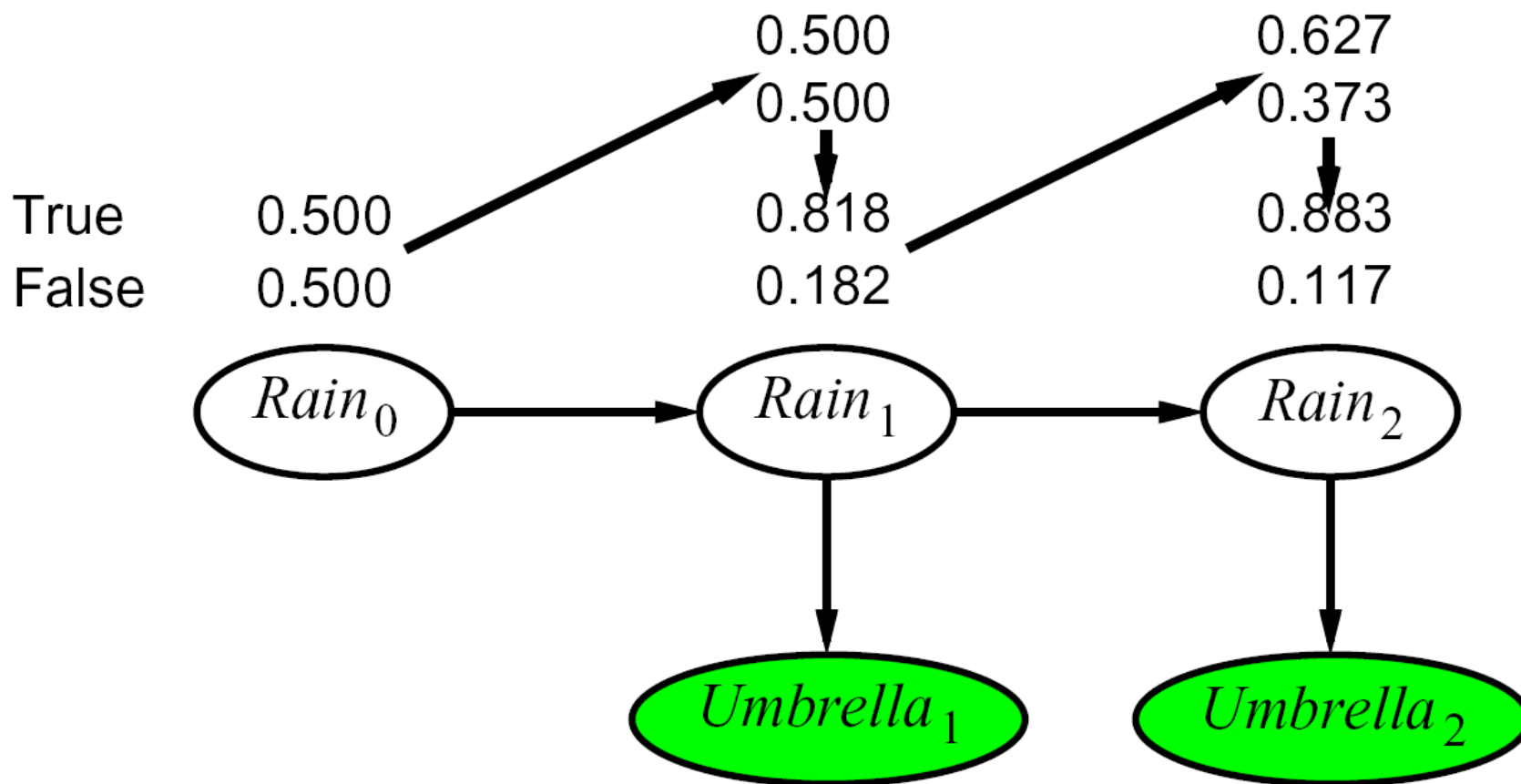
Before observation



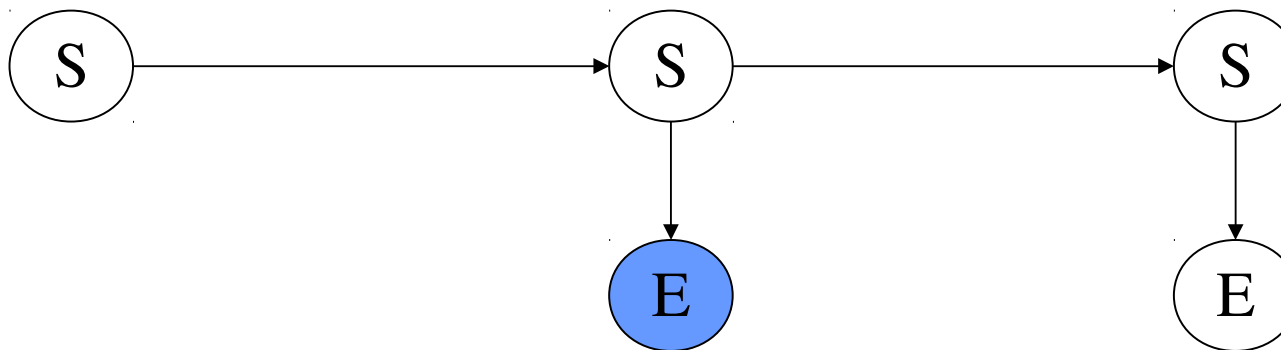
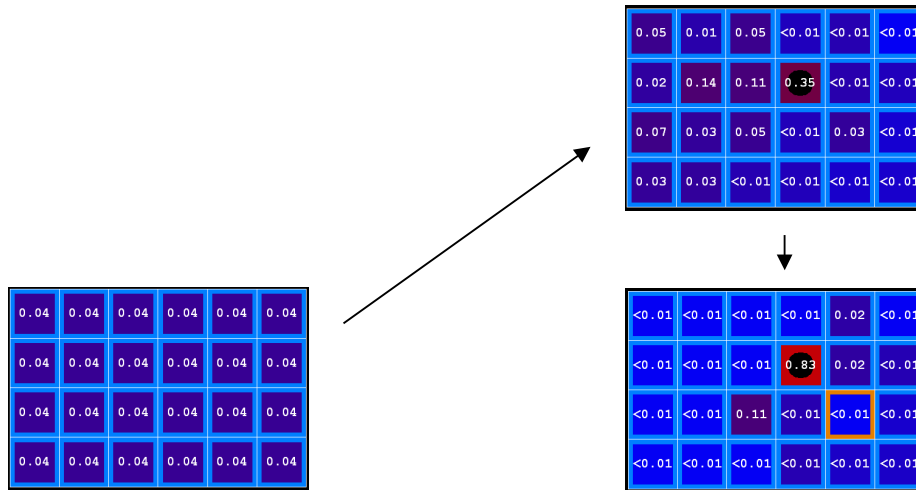
After observation

$$B(X) \propto P(e|X)B'(X)$$

# Example HMM



# Example HMM



# Updates: Time Complexity

- Every time step, we start with current  $P(X \mid \text{evidence})$
- We must update for time:

$$P(X_t | e_{1:t-1}) \propto \sum_{x_{t-1}} P(X_t | x_{t-1}) P(x_{t-1} | e_{1:t-1})$$

- We must update for observation:

$$P(X_t | e_{1:t}) \propto P(e_t | X_t) P(X_t | e_{1:t-1})$$

- So, linear in time steps, quadratic in number of states  $|X|$
- Of course, can do both at once, too

# The Forward Algorithm

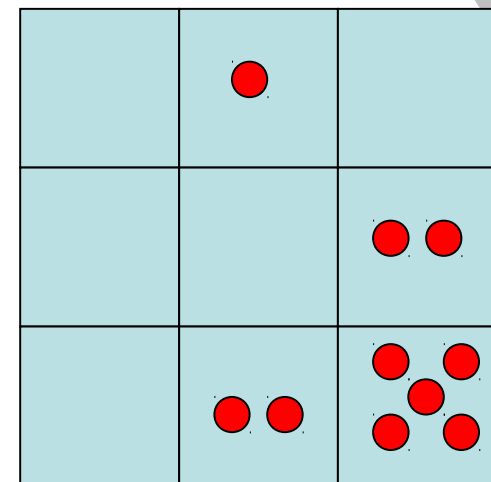
- Can do belief propagation exactly as in previous slides, renormalizing each time step
- In the standard forward algorithm, we actually calculate  $P(X, e)$ , without normalizing (it's a special case of VE)

$$\begin{aligned} P(x_t | e_{1:t}) &\propto P(x_t, e_{1:t}) \\ &= \sum_{x_{t-1}} P(x_{t-1}, x_t, e_{1:t}) \\ &= \sum_{x_{t-1}} P(x_{t-1}, e_{1:t-1}) P(x_t | x_{t-1}) P(e_t | x_t) \\ &= P(e_t | x_t) \sum_{x_{t-1}} P(x_t | x_{t-1}) P(x_{t-1}, e_{1:t-1}) \end{aligned}$$

# Particle Filtering

- Sometimes  $|X|$  is too big to use exact inference
  - $|X|$  may be too big to even store  $B(X)$
  - E.g.  $X$  is continuous
  - $|X|^2$  may be too big to do updates
- Solution: approximate inference
  - Track samples of  $X$ , not all values
  - Time per step is linear in the number of samples
  - But: number needed may be large
- This is how robot localization works in practice

0.0	0.1	0.0
0.0	0.0	0.2
0.0	0.2	0.5

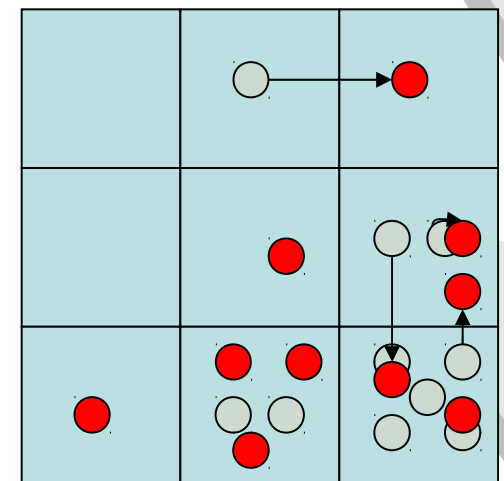
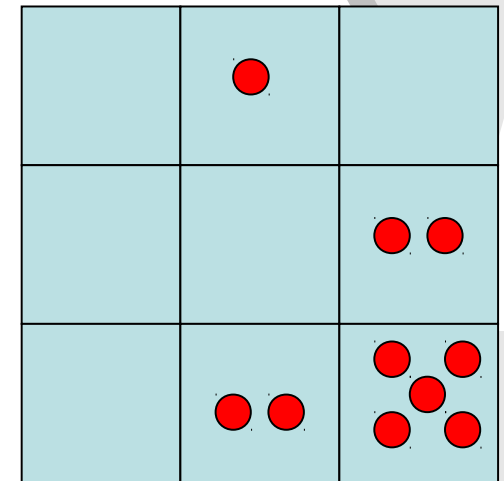


# Particle Filtering: Time

- Each particle is moved by sampling its next position from the transition model

$$x' = \text{sample}(P(X'|x))$$

- This is like prior sampling – samples are their own weights
- Here, most samples move clockwise, but some move in another direction or stay in place
- This captures the passage of time
  - If we have enough samples, close to the exact values before and after (consistent)



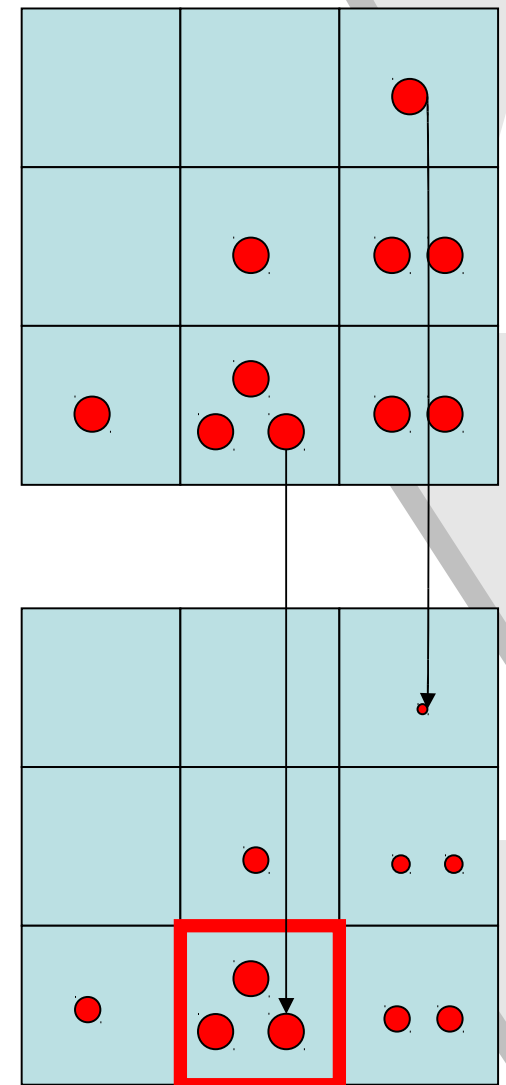
# Particle Filtering: Observation

- Slightly trickier:
  - We don't sample the observation, we fix it
  - This is similar to likelihood weighting, so we downweight our samples based on the evidence

$$w(x) = P(e|x)$$

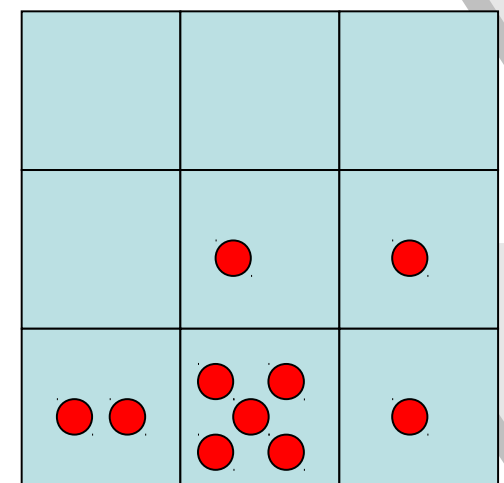
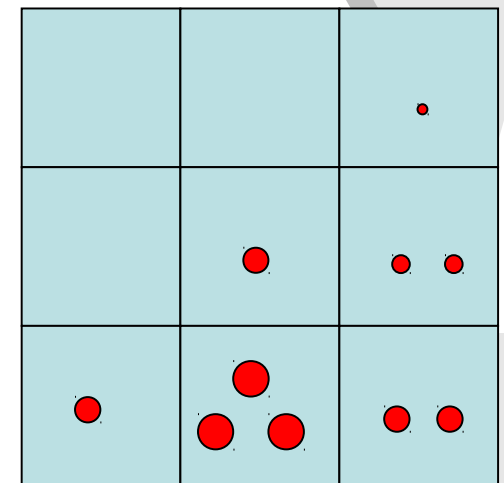
$$B(X) \propto P(e|X)B'(X)$$

- Note that, as before, the probabilities don't sum to one, since most have been downweighted (they sum to an approximation of  $P(e)$ )



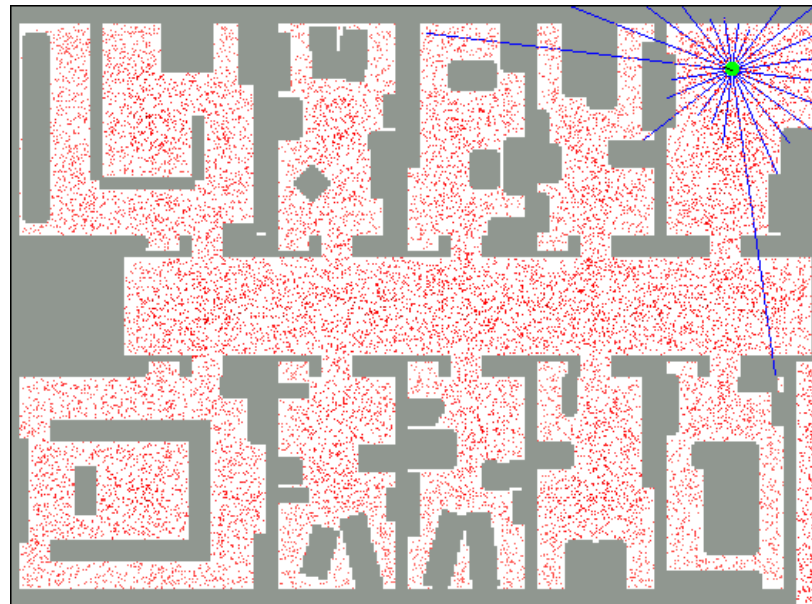
# Particle Filtering: Resampling

- Rather than tracking weighted samples, we resample
- N times, we choose from our weighted sample distribution (i.e. draw with replacement)
- This is equivalent to renormalizing the distribution
- Now the update is complete for this time step, continue with the next one



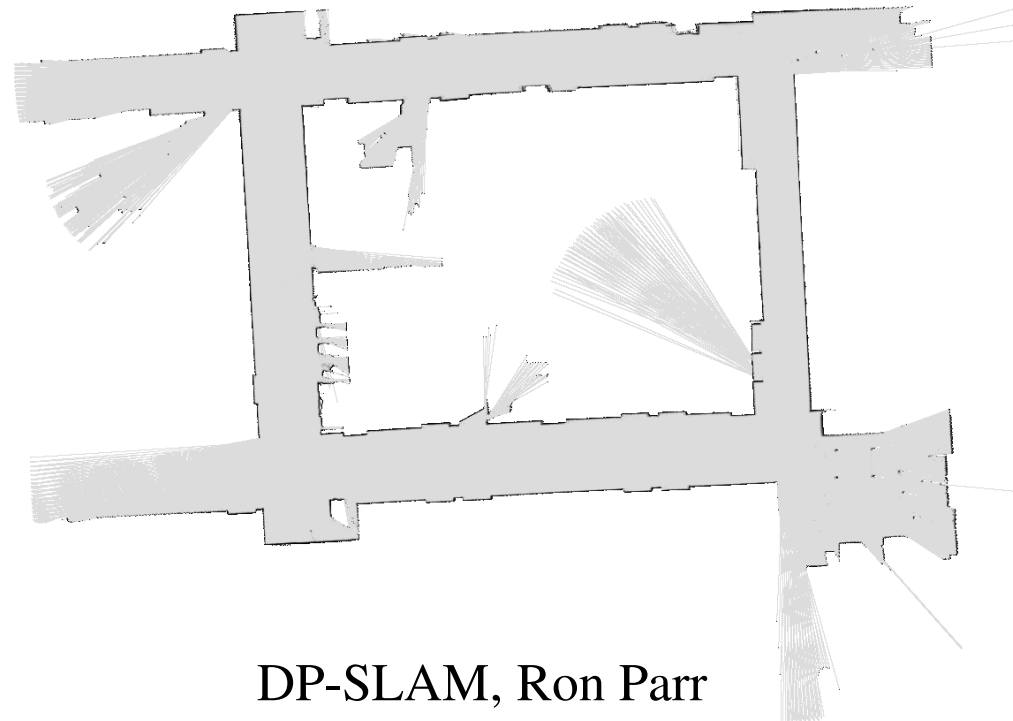
# Robot Localization

- In robot localization:
  - We know the map, but not the robot's position
  - Observations may be vectors of range finder readings
  - State space and readings are typically continuous (works basically like a very fine grid) and so we cannot store  $B(X)$
  - Particle filtering is a main technique



# SLAM

- SLAM = Simultaneous Localization And Mapping
- We do not know the map or our location
- Our belief state is over maps and positions!
- Main techniques: Kalman filtering (Gaussian HMMs) and particle methods



DP-SLAM, Ron Parr

# Most Likely Explanation

- Question: most likely sequence ending in  $x$  at  $t$ ?
  - E.g. if sun on day 4, what's the most likely sequence?
  - Intuitively: probably sun all four days
- Slow answer: enumerate and score

$$P(X_t = sun) = \max_{x_1 \dots x_{t-1}} P(x_1, \dots, x_{t-1}, sun)$$

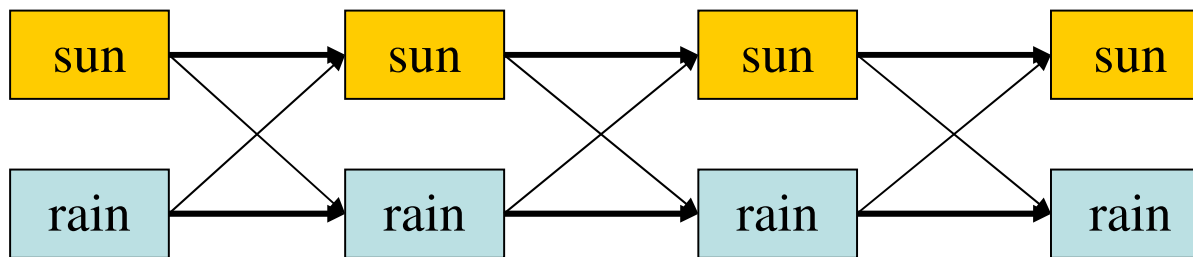
$$P(X_1 = sun)P(X_2 = sun|X_1 = sun)P(X_3 = sun|X_2 = sun)P(X_4 = sun|X_3 = sun)$$

$$P(X_1 = sun)P(X_2 = rain|X_1 = sun)P(X_3 = sun|X_2 = rain)P(X_4 = sun|X_3 = sun)$$

⋮

# Mini-Viterbi Algorithm

- Better answer: cached incremental updates

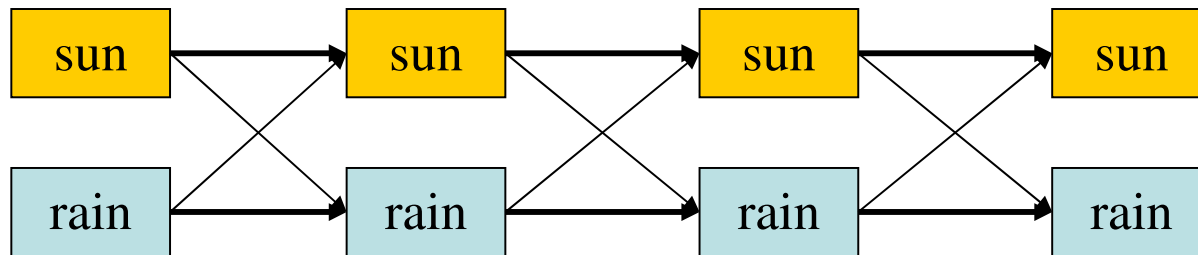


- Define:
 
$$m_t[x] = \max_{x_{1:t-1}} P(x_{1:t-1}, x)$$

$$a_t[x] = \arg \max_{x_{1:t-1}} P(x_{1:t-1}, x)$$

- Read best sequence off of m and a vectors

# Mini-Viterbi



$$m_t[x] = \max_{x_{1:t-1}} P(x_{1:t-1}, x)$$

$$= \max_{x_{1:t-1}} P(x_{1:t-1})P(x|x_{t-1})$$

$$= \max_{x_{t-1}} P(x_t|x_{t-1}) \max_{x_{1:t-2}} P(x_{1:t-1})$$

$$= \max_{x_{t-1}} P(x_t|x_{t-1})m_{t-1}[x]$$

$$m_1[x] = P(x_1)$$

# Viterbi Algorithm

- Question: what is the most likely state sequence given the observations?
- Slow answer: enumerate all possibilities
- Better answer: cached incremental version

$$x_{1:T}^* = \arg \max_{x_{1:T}} P(x_{1:T} | e_{1:T})$$

$$m_t[x_t] = \max_{x_{1:t-1}} P(x_{1:t-1}, x_t, e_{1:t})$$

$$= \max_{x_{1:t-1}} P(x_{1:t-1}, e_{1:t-1}) P(x_t | x_{t-1}) P(e_t | x_t)$$

$$= P(e_t | x_t) \max_{x_{t-1}} P(x_t | x_{t-1}) \max_{x_{1:t-2}} P(x_{1:t-1}, e_{1:t-1})$$

$$= P(e_t | x_t) \max_{x_{t-1}} P(x_t | x_{t-1}) m_{t-1}[x_{t-1}]$$

# Example

