Surface Reconstruction with MLS

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Literature

• *An Adaptive MLS Surface for Reconstruction with Guarantees*, T. K. Dey and J. Sun

• *A Sampling Theorem for MLS Surfaces*, Peer-Timo Bremer, John C. Hart
An Adaptive MLS Surface for Reconstruction with Guarantee

Tamil K. Dey and Jian Sun
Implicit MLS Surfaces

\[ I(x) = \frac{\sum_{p \in P} [(x - p)^T v_p] \theta_p(x)}{\sum_{p \in P} \theta_p(x)} \]

- Zero-level-set of function \( I(x) \) defines \( S \)
  - Normals are important, because points are projected along the normals
Motivation

• Original smooth, closed surface $S$.

• Given Conditions:
  – Sampling Density
  – Normal Estimates
  – Noise

→ Design an implicit function $\mathcal{I}(x)$ whose zero set recovers $S$. 
Motivation

- So far, only uniform sampling condition.
- Restriction:
  - The red arc requires $10^4$ samples because of small feature.
Motivation

• Establish an *adaptive* sampling condition, similar to the one of Amenta and Bern.

→ Incorporate local feature size in sampling condition.
  • Red arc only requires 6 samples.
Sampling Condition

• Recent Surface Reconstruction algorithms are based on noise free samples. 
  → Notion of $\varepsilon$-sample has to be modified.

• $P$ is a noisy $(\varepsilon, \alpha)$-sample if
  – Every $z \in S$ to its closest point in $P$ is $< \varepsilon \text{lfs}(z)$.
  – The distance for $p \in P$ to $z = \text{proj}(p)$ is $< \varepsilon^2 \text{lfs}(z)$.
  – Every $p \in P$ has a normal which has an angle with its corresponding projected surface point $< \varepsilon$.
  – The number of sample points inside $B(x, \varepsilon \text{lfs}(\text{proj}(x)))$ is less than a small number $\alpha$.

→ Note that it is difficult to check whether a sample $P$ is a noisy $(\varepsilon, \alpha)$-sample.
Effect of nearby samples

• Points within a small neighborhood are predictably distributed within a small slab:

**Lemma 1** For $\rho \leq 1$ and $\varepsilon \leq 0.1$, any sample point inside $B(z, \rho lfs(z))$ lies inside the slab bounded by the planes $PL_+$ and $PL_-$ where

\[
\omega = \frac{(\varepsilon^2 + \rho)^2}{2(1 - \varepsilon^2)^2} + \frac{(1 + \rho)}{1 - \varepsilon^2} \varepsilon^2.
\]
Adaptive MLS

• \( \mathcal{J}(x) \) at point \( x \) should be decided primarily by near sample points.
  → Choose \( \theta_p(x) \) such, that sample points outside a neighborhood have less effect.
  → Use Gaussian functions.
  → Their width control influence of samples.

• Make width dependent on \( lfs \)
  → Define width as fraction of \( lfs \)
Adaptive MLS

\[ \mathcal{I}(x) = \frac{\sum_{p \in P}[(x - p)^T v_p] \theta_p(x)}{\sum_{p \in P} \theta_p(x)} \]

- Choice of weighting function:

\[ \theta_p(x) = e^{-\frac{||x-p||^2}{[\rho \text{lfs}(\tilde{x})]^2}} \]

→ Many sample points at \( p \) which contribute to point \( x \).
Adaptive MLS

\[ I(x) = \frac{\sum_{p \in P}[(x - p)^T w_p] \theta_p(x)}{\sum_{p \in P} \theta_p(x)} \]

- Choice of weighting function:

\[ \theta(x)_p = e^{-\frac{\|x - p\|^2}{[\rho lfs(\tilde{p})]^2}} \]

→ Sample point \( p \) has a constant weight(\( e^{-\frac{1}{[\rho \cos \beta]^2}} \)).

→ Influence of \( p \) does not decrease with distance.
Adaptive MLS

• Compromise: Take fraction of $\sqrt{\text{lfs}(\tilde{x})\text{lfs}(\tilde{p})}$ as width of Gaussian.

→ Weighting function decreases as $p$ goes far away from $x$.
→ Weighting is small for small features, i.e. small features do not require more samples.
Contribution of distant samples

• Show that the effect of distant sample points can be bounded.
  → Rely on nearby features.

• How to show that?
Contribution of distant samples

Lemma 2 For $\rho \leq 0.4$ and $\varepsilon \leq 0.1$, the number of sample points inside $B(x, \frac{\rho}{2}\text{lfs}(\bar{x}))$ is less than $\lambda$ where

$$\lambda = \alpha$$

if $\rho \leq 2\varepsilon$

$$= \frac{75\rho^3 \alpha}{\varepsilon^3}$$

otherwise.
Contribution of distant samples

- Once there is a bound on the number of samples in $B_{\rho/2}$, we lemma 3 is used which shows an upper bound on its influence, i.e.

**Lemma 3** If $\rho \leq 0.4$, $\varepsilon \leq 0.1$ and $r \geq 5\rho$,

$$
\sum_{p \in B_{\rho/2} \cap S_x(w_i, \rho)} I_p(x) \leq \lambda e^{-\frac{r w_i}{(1+2r)\rho^2}} \cdot \frac{W_i^s}{\rho^{2t}} f(\tilde{x})^{s-2t}
$$
Contribution of distant samples

• Using lemma 3 they prove the main theorem:

**Theorem 1** If $\rho \leq 0.4$, $\varepsilon \leq 0.1$ and $r \geq 5\rho$, then for any $x \in \mathbb{R}^3$

$$\sum_{\rho \notin B(x,rf(\bar{x}))} I_p(x) \leq C_1 \lambda \cdot \frac{r^2 + r\rho + \rho^2}{\rho^2} e^{-\frac{r^2}{(1+2r)\rho^2}} \cdot \frac{r^s}{\rho^{2t}} f(\bar{x})^{s-2t}$$
Contribution of distant samples

- The space outside $B(x, \text{rlfs}(%20\text{proj}(x)))$ can be decomposed in an infinite number of shells, i.e.

$$
\sum_{p \notin B(x, rf(x))} I_p(x) = \sum_{i=0}^{\infty} \sum_{p \in S_x(w_i, \rho)} I_p(x)
$$

→ The influence of all points outside is equal to the influence of all the points in the shells which was bounded by lemma 3.

→ Therefore, the contributions of points outside $B(x, \text{rlfs}(%20\text{proj}(x)))$ can be bounded.
Algorithm

AMLS(P)

**NORMAL ESTIMATION:**
Compute Del(P)
for each point p with big Delaunay ball
compute the normal n_p

**FEATURE ESTIMATION:**
for each p ∈ P estimate f(\tilde{p})

**PROJECTION:**
for each p ∈ P
project p to p' by Newton iteration;
if ||p − p'|| > \tau go to 1 with p := p'
endfor

**RECONSTRUCTION:**
Let P' be the projected point set;
reconstruct with P'.
Normal Estimation

• Delaunay based, i.e. calculate DT of input points.

• Big Delaunay ball: radius greater than certain times the nearest neighbor.

• The vectors of incident points to center of $B(c,r)$ approximate normals.
Normal Estimation
Feature Detection

• In noiseless case:
  → Take poles in Voronoi cells, shortest distance approximates $lfs(p)$.

• This does not work in the case of noise.
• Every medial axis point is covered with a big Delaunay Ball (observation by Dey and Goswami)
  • Take biggest Delaunay Ball in both (normal) directions of sample point $p$.
    → Centers act as poles ($L$).
• $lfs(proj(x))$ is $d(p, L)$ where $p$ is closest point to $x$ in $P$. 
Projection

• Newton Iteration: move \( p \) to \( p' \), i.e.:

\[
p' = p - \frac{\mathcal{I}(p)}{\| \nabla \mathcal{I}(p) \|^2} \nabla \mathcal{I}(p)
\]

• Iterate until \( d(p, p') \) becomes smaller than a certain treshold.

• To calculate \( \mathcal{I}(p) \) and \( \nabla \mathcal{I}(p) \) only use the points in Ball with radius 5x the width of the Gaussian weighting function.

\[\rightarrow\] Points outside have little effect on the function.

• Newton projection has big convergent domain.
Reconstruction

• Finally, use a reconstruction algorithm, e.g. Cocone to calculate a mesh.
AMLS vs PMLS

• Background on “Projection MLS”:
  – Surface is the set of stationary points of a function \( f \).
  – An energy function \( e \) measures the quality of the fit of a plane to the point set at \( r \).
  – The local minima of \( e \) nearest to \( r \) is \( f(r) \).
AMLS vs PMLS

• The zero set of the energy function defines the surface.

• But there are other two layers of zero-level sets where the energy function reaches its local maximum.

\[ \varepsilon(y, n(x)) = \frac{1}{2} \sum_{p \in P} [(y - p)^T n(x)]^2 \theta_p(y) \]

\[ J(x) = n(x)^T \left( \frac{\partial \varepsilon(y, n(x))}{\partial y} \right|_x \)
AMLS vs PMLS

• When distance between the layers gets small, computations on the PMLS surface become difficult.
  → Small “marching step” in ray tracing
  → Small size of cubes in polygonizer
AMLs vs PMLS

- Furthermore, projection procedure in PMLS is not trivial.
  → non-linear optimization.
  → ...and finding a starting value is difficult if the two maximum layers are close.

- If there is noise, maxima layers could interfere.
  → Holes and disconnectness.
# AMLS vs PMLS

| Model       | \( |P| \)   | Method | \#nb | \#iter | Time |
|-------------|--------|--------|------|--------|------|
| Max-planck  | 49137  | NP     | 1000 | 3.1    | 94   |
|             |        | PP     | 1108 | 7.2    | 310  |
| Bighand     | 38214  | NP     | 1392 | 3.2    | 109  |
|             |        | PP     | 1527 | 8.6    | 400  |
A Sampling Theorem for MLS Surfaces

Peer-Timo Bremer and John C. Hart
Motivation

• We saw a lot of work about PSS.

• But not much knowledge about resulting mathematical properties.

• It is assumed that surface construction is well defined within a neighborhood.
Motivation

• MLS filters samples, and projects them onto a local tangent plane.

• MLS is robust but difficult to analyze.

• Current algorithms may actually miss the surface.
  → MLS is defined as the stationary points of a (dynamic) projection.
  → This projection is “dynamic” and can be undefined.
  → Robustness of MLS is determined by projection.

→ Need a sample condition which guarantees that projection is defined everywhere.
Motivation

• What is necessary to have a faithful projection?
  → Correct normals.

• We need a condition which shows that given a surface and a sampling, normals are well defined.
Sampling Condition - Outline

• Given a sample, show

  Normal vector needed by MLS does not vanish

  Given conditions of the surface and sampling, the normal is well defined
PCA - Review

- We have given a set of points in $\mathbb{R}^3$ which lie or don’t lie on a surface.
- We know the 3x3 covariance matrix at point $q$ is

$$
cov_\theta(q) = \sum_i \theta(||q - p_i||)(q - p_i)(q - p_i)^T
$$

- The $cov(q)$ can be factored into $U^TLU$

- ...where

$$
L = \begin{pmatrix}
\lambda_{\text{max}} & 0 & 0 \\
0 & \lambda_{\text{mid}} & 0 \\
0 & 0 & \lambda_{\text{min}}
\end{pmatrix}
$$

$$
U = \begin{pmatrix}
v_{\text{max}} & v_{\text{mid}} & v_{\text{min}}
\end{pmatrix}
$$
PCA - Review

- $v_{\text{max}}$, $v_{\text{mid}}$, $v_{\text{min}}$ define local coordinate frame:
MLS Surface

- *weight function* $\theta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ e.g. Gaussian

- Weighted average
  \[
  a(q) = \frac{\sum_i \theta(||q - p_i||)p_i}{\sum_i \theta(||q - p_i||)}
  \]

- 3x3 covariance matrix
  \[
  \text{cov}_\theta(q) = \sum_i \theta(||q - p_i||)(q - p_i)(q - p_i)^T
  \]
MLS surface

- From $\text{cov}_\theta(q)$ we know:
  The eigenvector of the unique smallest eigenvalue defines the normal $n(q)$.

- Using that, the zero set of

  $$f(q) = n(q)^T (q - a(q))$$

defines the MLS surface.

  $\Rightarrow n(q)$ has to be well defined!
New Sampling Condition

• Previous Sampling condition:
  “S is well sampled by \( P \) if normal directions are defined inside a neighborhood of \( \hat{S} \).” (Adamson and Alexa)

• This is not good because \( S \neq \hat{S} \).
  \( \rightarrow \) S could be “well sampled” but undefined normal direction can result

• New Sampling condition:
  “S is well sampled by \( P \) if normal directions are defined inside a neighborhood of S.
Uniqueness of smallest Eigenvalue

• You can prove a sampling to be well defined by ensuring that \( \text{cov}_\theta(q) \) has a unique smallest eigenvalue over a neighborhood of \( S \).

• It is proved in terms of the weighted variance of the samples \( P \).

• Directional Variance:

\[
\text{var}_n(q) = \sum_{i=1}^{N} \theta(||q - p_i||) (n^T (q - p_i))^2
\]

→ Variance in a specific direction
Uniqueness of smallest Eigenvalue

• We combine that with $\text{cov}_\theta(q)$ and get

$$\text{var}_n(q) = n^T \text{cov}_\theta(q)n$$

• …and decompose it into eigenvalues and eigenvectors:

$$\text{var}_n(q) = a^2 \lambda_1 + b^2 \lambda_2 + c^2 \lambda_3$$

where $a = n^T v_1$, $b = n^T v_2$ and $c = n^T v_3$
Uniqueness of smallest Eigenvalue

• Using that it can be shown that if $\lambda_{\text{min}}$ is not unique $\rightarrow \text{var}_n(q)$ isn’t unique either. (Lemma 2)

• This leads to:
Theorem 1 If the directional weighted variance $\text{var}_v(q)$ for some unit vector $v$, is strictly less than the directional weighted variance of any perpendicular unit vector $w$, then the smallest eigenvalue of $\text{cov}_\theta(q)$ is unique.
Sampling Theorem

• To prove the sampling condition, show that for every point $q$, there is a normal direction whose weighted variance is less than that of any perpendicular direction.
  → Derive *upper* bound of weighted variance in normal direction $n$.

→ Derive *lower* bound in an arbitrary tangent direction $x$.

→ Determine sampling conditions:
  $\max(var_n(q)) < \min(var_x(q))$
Sampling Theorem

• To show $\text{var}_n(q) < \text{var}_x(q)$, partition datapoints into ones that increase $\text{var}_n$ more versus ones that increase $\text{var}_x$ more.

• Construct two planes through $q$

$$P_{\pm} = \{ p \in \mathbb{R}^3 \mid n^T (p - q) = \pm x^T (p - q) \}$$

...which separates the points such that
Sampling Theorem

- Points below those planes increase $\text{var}_n(q)$ more than $\text{var}_x(q)$.
- The region below the planes define an *hourglass* shape.
Sampling Theorem

• Furthermore, $q$ is at most $w = \tau lfs(proj(q))$ above surface.
Sampling Theorem

• Project hourglass onto lower medial ball (B-).
• Find limit of max possible effect of these “risky” samples.
• Use the risky area (red) to overestimate the number of samples and their contribution to \( var_n \).
Sampling Theorem

• Define upper bound on samples in risky area and use it to compute upper bound of $var_n$.

• Define lower bound on samples outside risky area to compute lower bound of $var_x$.

• This leads to the Main Theorem:

**Theorem 2** Let $S$ be a surface of bounded positive local feature size $\rho_{\text{max}}/\rho_{\text{min}} \leq \alpha$, sampled by an $(\varepsilon, \delta)$-sampling of points $p \in P$ no farther than $\tau \rho(w)$ from $S$, where $w$ is the closest point on $S$ to $p$. Then for $\alpha \leq 1000, \varepsilon \leq 1/200, \delta \geq 1/2000$, and $\tau \leq 1/250$, the MLS surface constructed with an adaptive Gaussian kernel of standard deviation $\sigma = \rho(w)/25$ on the samples $P$ is well defined in that its normals never vanish over the $\tau$ neighborhood of $S$. 