Theory of Hexahedral Mesh Generation

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Outline

• Review
• Hexahedral Quality
• Why is hex meshing hard?
• Plastering
• Topology theory
• Whisker Weaving
• Introduction to my research…
Hexes

Structured & Semi-structured
- Mapping
- Submapping
- Primitives
- Sweeping

Unstructured
- Direct
  - Grid-Based
  - Plastering
- Indirect
  - H-Morph
  - THex
- ...
What scheme?
What scheme?
What scheme?
What scheme?
What scheme?

*Model created by Andy Anderson, University of Utah

*Model created by Alex Smith, University of New Mexico
What scheme?
Hexahedral Quality

• How to measure good quality?
  – Review FEA governing equations…

• The Jacobian of a hex…

• Undesireable characteristics in hexes…

• Shape improvement measures…
  – Smoothing, optimization, and untangling…
• Can anyone create an all-hex mesh conforming to the following surface mesh?
  • [http://www-users.informatik.rwth-aachen.de/~roberts/open.html](http://www-users.informatik.rwth-aachen.de/~roberts/open.html)

  – He didn’t really say why you’d want to. But, if you could, then for any model, you could Plaster a while, add these, fill with tets, then divide into all-hex mesh.
  – Are there “bad” surface meshes that aren’t fillable?
Plastering

Remaining Surface Faces

Remaining Void

After Hex Generation
Plastering

• **Kind of worked**
  – Could fill up most of space

*Figure 1: Winblock Model and Void Remaining after Hex Generation*
Why is Hex Meshing Hard?

• Connectivity of tet meshing is “easy”
  – Delaunay triangulations
    • Given set of points, can connect them up to form triangles, tetrahedra, d-simplices
    • Empty-sphere property shows you how
  • Caveats
    – Good quality is difficult in 3 dimensions
    – Boundary constraints is difficult in 3 dimensions
  – Octree (subdivision) techniques
    • Keep dividing polyhedron until you get simplices
      – No new points on the boundary are necessary
  – Advancing front techniques
    • Remaining space always dividable into tetrahedra
Why is Hex Meshing Hard?

- Quads harder than tris (i.e. even 2d is harder)
  - Can’t just divide polygons into quads
  - New points on the boundary may be necessary

In practice, an even number of boundary edges can be divided into quads (with new interior points).
- Necessary and sufficient
  
Euler: $4q = 2e_{int} + e_{bdy}$
Voronoi Diagrams

Voronoi region:

\[ V(p_i) = \{ x \in \mathbb{R}^n | \|p_i - x\| \leq \|p_j - x\|, \forall j \leq n \} \]
Delaunay Triangulation

- Dual to Voronoi Diagram
- Connect Vertices Across Common Line

*From Solomon Boulos slides earlier in the semester....
Dual of a hexahedral mesh

• Assume that a quad mesh represents the Voronoi diagram…
• The dual to the quad mesh has some interesting properties…
Opposite pairs of edges on a quadrilateral are connected with a line segment.
A column of elements is represented by a single line (known as a chord).
The Dual of a Hexahedral Mesh (2D)

- Quadrilaterals are formed by the intersection of two chords
- A quadrilateral is dual to the intersection point (known as a centroid)
• Chords either form closed loops, or exit at the boundary.
• Chords intersect nearly orthogonally with other chords.
• Chords cannot be tangent to other chords.
The Dual of a Hexahedral Mesh (2D)

- Chords either form closed loops, or exit at the boundary.
- Chords intersect nearly orthogonally with other chords.
- Chords cannot be tangent to other chords.

(Important observation: For a closed manifold [or closed set of surfaces] all chords form closed loops…)
The Dual of a Hexahedral Mesh (2D)

- Only two chords may intersect at any one centroid.

- **NOTE:** What do the chords look like on a closed manifold?
• In 3D, the chords become surfaces (known as sheets)
• Each sheet represents a layer of hexahedra
• The intersection of three sheets is dual to a hexahedron.
• No more than three sheets can intersect at a single point.
• Sheets cannot be tangent to other sheets.
• All sheets intersect nearly orthogonally with any other sheet.
• All sheets form either a closed shell in the space, or have terminating edges around the boundary of the space.
• **Thurston’s conjecture**
  – Given a closed set of surfaces and a quadrilateral mesh on those surfaces. The dual of the quadrilateral mesh is a closed set of curves which will be the boundaries of the interior surfaces of an enclosed hexahedral mesh

“This gives a way to construct the surface with boundary the given curves. It may not be dual to a hexahedral decomposition, because for instance there may be no interior vertices (corresponding to hexahedra) at all. So, add a bunch of spheres with enough intersection points with the existing surface to make lots of interior vertices. This turns it into the dual of a hexahedral decomposition.“

• Mitchell later formulated a proof of this idea
• Eppstein showed that hexahedral mesh generation is solvable with linear complexity (with respect to the number of quads on the boundary…)
Existence Proofs

Mitchell’s proof, very similar to Thurston’s outline, a couple years later (mid 1995)

- Map surface mesh to a sphere (smooth)
- Form STC loops – smooth closed curves
- Extend loops into closed surfaces
- Fix arrangement of surfaces to avoid degenerate elements
- Dualize to form hexes
- Map back to original object.
**Necessary**: surface mesh must have even #quads

**Pf**: Every hex mesh has an even #quads on surface:

\[ 6h = 2f - b \]

- \( h \) = #hexes
- \( f \) = total #faces
- \( b \) = #surface faces

(count faces)
Sufficient Conditions

• Every pair of loops on sphere intersect each other even number of times

• A single loop can self-intersect an even or odd number of times

• Topology theorems
  – For any single curve with even # self-intersections, can construct a surface (see Suzuki, et al.)
  – For any pair of curves with odd # self-intersections, can construct a surface
Main ideas of state list Whisker Weaving

- Start with fixed surface mesh
- This defines “boundary loops” of the sheets = twist-planes

```
green  red  blue  purple
```

```
1  2  3  4
5  6  7  8
9 10 11 12
13 14 15 16
17 18 19 20
21 22 23 24
25 26 27 28
29 30 31 32
33
```
Whisker Weaving #1 - State List

- Add dual of hex (STC vertex) from bdy inward, one at a time
  * Compare to plastering…
  * Each sheet is 2d: did we reduce this 3d problem to a set of 2d problems? I think not…although it is easier to understand and visualize than free-floating primal edges.

crossing = weaving
• Now it’s different than Plastering
  – Try to get good local connectivity
    • Connectivity Rules about where to add, when to seam, …
      – Easily identify seaming = joining cases
      – Rarely join two loops together
        • Keep sheets as disk
  – Geometric rules: consider geometric dihedral angles of faces near surface mesh, no flat hexes
  – Heroic effort by Tautges to build rules for good all-hexes and close the weave
    • In practice, often unclosed region left-over, like Plastering
    • In practice, very complicated sheets
WW complicated sheets

- One or two sheets going through most of the surface quads
- Related to expected connectivity of random graphs on lattice?
- Red = knives (wedges)
Whisker Weave #2 - Curve Contraction

- Implemented the existence proof for curves w/out self-intersections, sphere surfaces. (Nate Folwell, Mitchell)
  - To get surface, take loop and extend topologically next to the remaining void
  - Pre-process surface mesh to get rid of self intersections.
- Face collapse and refinement.
- Always solves the topology problem (existence proof is a proof, after all...), but rarely the geometric problem
Self-Intersection Removal

- Face collapse

- and pillowing
Whisker Weaving #2 – Curve Contraction

Self-Intersection Removal

• Bad surface mesh quality in practice
Whisker Weaving #3 – Dual Cycle Elimination

• Matthias Muller-Hannemann
• Similar to curve contraction algorithm
• Advantages
  – No fix-up, instead prevents bad cases
    • Sufficient conditions to eliminate a cycle without introducing degenerate or flat elements
    • (but sometimes not possible to make progress)
  – No self-intersecting curves for arrangements
    • German CAD tool geometry decomposer into 4-sided surfaces, and well structured surface mesher
    • (NOTE: Possible relationship to Quadrangulation research discussed by Carlos earlier…)}
Whisker Weaving #3 – Dual Cycle Elimination

- **Basic principle**
  - Cycles eliminated one by one
  - *As WW curve contraction*
Whisker Weaving #3 – Dual Cycle Elimination

• Remove self-intersecting loops ahead of time
  – Dice one level
    • High element count
  – Replace self-intersections with template
    • Good quality compared to WW approach
  – Works for assembly with shared surfaces
    • Doesn’t introduce new self-intersections
Whisker Weaving #3 – Dual Cycle Elimination

• Application success
  – Human mandible
    • Good quality hex mesh
    • Had to do some manual work to identify good surface mesh.
Whisker Weaving #4 – Recursive Bisection

- Calvo and Idelsohn in Argentina
- Select a cycle for elimination = insert a layer of hexes
  - Instead of layer next to boundary, it cuts the model in two.
    - Layer has connectivity of one of the sides
    - Maybe better geometrically?
  - Two halves resolved independently
    - Topological “monsters” (degeneracies) as WW
Hexahedral Mesh Generation Constraints

• Mitchell’s proof (and Curve-Contraction Whisker Weaving) delineate the topologic constraints needed for hexahedral mesh generation.
  – The Whisker Weaving algorithms never were able to guarantee an acceptable quality hexahedral mesh.
  – Something missing?

• Some additional constraints are obviously needed to have provably correct hexahedral mesh generation
  – Geometric Constraints?
  – Quality Constraints?
Schneiders Open Problem Closed?

• Using these kind of techniques, a dozen people have constructed a “mesh” for the open problem.
  – One self-intersecting curve, 8 times. 2 curves.
  – Non-degenerate meshes have > 100 hexes (Poor quality)

• Yamakawa has produced a mesh of the pyramid with positive jacobians in 88 hexes…
• Suzuki, et al. produced a symmetric solution with 156 hexes (haven’t resolved negative jacobians…)
• The smallest hexahedral topology produced to date (by Suzuki, et al.) which fills the pyramid has 20 hexes (degenerate)…
Current Research Activity

• Flipping (Bern, et al., Tautges)
  – Find areas of poor quality and apply small set of templates to ‘flip’ elements to a new state with improved quality
• Unconstrained Plastering (Staten, et al.)
  – Advancing front sheet insertion
• Direct sheet insertion (Suzuki, Takahashi, Shepherd)
  – Geometric creation of sheets from boundary cycles
• Untangling and Optimization (Knupp, Freitag, et al.)
  – Move nodal locations to locate an optimized/untangled mesh state
• Constrained methods (Carbonera, Shepherd)
  – Provides a numerical constant to Eppstein’s linear complexity proof (paper should be forthcoming in this year’s IMR)
• Hexahedral Constraints (Shepherd)
  – Delineate all necessary constraints to fill an arbitrary space with hexahedral elements
Pillowing

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A

B

C
Dicing
Sheet Extraction

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Refinement and Coarsening

Boundary Refinement

Sheet Refinement & Cleaving

Mesh Coarsening
Grafting
Sheet Insertion

• Modify mesh to capture geometry, through STC sheets
The problem is to fit the mesh on the left to the volume on the right. The rest of the volumes in the model can then be swept away from this volume.
Step 1: Capturing the first imprint.
Step 2: Capturing the other two imprints

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Step 3: Capturing the Cutouts