Homework 2 (Due Thursday, Feb 4 (23:59 MT))

Policy: You are allowed to reference any material, but please cite the source and write the solutions in your own words. Discussing solutions is OK, as long as you mention the names of students you discussed with.

1. (10 points) A 3-SAT formula on \( n \) variables with \( m \) clauses is the following: we have \( n \) boolean variables \( x_1, x_2, \ldots, x_n \), and \( m \) clauses \( C_1, C_2, \ldots, C_m \). Each \( C_i \) is the boolean OR of three distinct literals (a literal is just a variable \( x_i \) or its negation \( \neg x_i \)). An example of a clause is \( x_2 \lor x_4 \lor \neg x_5 \). An assignment \( \sigma \in \{T, F\}^n \) to the \( x_j \)'s is said to satisfy clause \( C_i \) if one of the literals in the clause is set to TRUE. (E.g., to satisfy the clause above, we must have either \( x_2 \) or \( x_4 \) to be TRUE, or \( x_5 \) to be FALSE.)

Show that for any 3-SAT formula with \( m \) clauses, there exists an assignment that satisfies at least \( \frac{7}{8}m \) clauses.

2. (10 points) A set system \( F \) over the set \([n] = \{1, 2, \ldots, n\}\) is a collection of subsets \( S_1, S_2, \ldots, S_m \) of \([n]\). Our goal is to color the integers \( \{1, 2, \ldots, n\} \) with two colors (say \(-1, 1\)), such that each of the sets \( S_i \) is as balanced as possible. Formally, if we have a coloring \( \chi : [n] \rightarrow \{-1, 1\} \), the imbalance (or discrepancy) is defined as
\[
\max_{1 \leq i \leq m} \sum_{j \in S_i} \chi(j).
\]

Prove that for any set system \( F \) with \( m \) sets, there exists a coloring with imbalance at most \( O(\sqrt{n \log m}) \).

3. (10 points) Let \( G \) be a connected \( d \)-regular graph, and consider its adjacency matrix \( A_G \). Prove that \( \lambda_{\min}(A_G) = -d \) if and only if \( G \) is bipartite.

4. (10 points) Let \( G \) be a connected \( d \)-regular graph, and suppose the Laplacian \( L_G \) has eigenvalues \( 0 \leq \lambda_2 \leq \cdots \leq \lambda_n \). Then, without using Cheeger’s inequality, prove that \( \lambda_2 \geq \frac{1}{4d} \).

HINT: recall that \( \lambda_2 = \min_{||x||=1} \sum_{i,j} x_i = 0 \sum_{(i,j) \in E} (x_i - x_j)^2 \); prove that there exists an \( x_u, x_v \) such that \( |x_u - x_v| \geq 1/\sqrt{n} \), and consider the path from \( u \) to \( v \).

5. (10 points) Let \( G \) be a \( d \)-regular graph with the property that every subset \( S \) of size at most \( n/2 \) has expansion at least \( 1/4 \), i.e.,
\[
\frac{E(S, V \setminus S)}{d|S|} \geq \frac{1}{4}.
\]

(a) (3 points) Define the neighborhood \( N(S) \) of a set \( S \) to be the set of all vertices \( u \in V \setminus S \) that have at least one edge to a vertex in \( S \). Prove that for any \( V \) of size at most \( n/2 \) in our graph \( G \),
\[
|N(S)| \geq |S|/4.
\]
(b) (7 points) Prove that the diameter of the graph is $O(\log n)$. I.e., for any $u, v \in V$, prove that there exists a path of length at most $O(\log n)$ between $u$ and $v$.

*HINT:* start with $u$, and consider the number of vertices at a distance $\ell$ from $u$; inductively do this for $\ell = 1, 2, \ldots$; for $\ell = O(\log n)$, argue that the number is $\geq n/2$; do the same with $v$. 