# Homework 1 (Due Thursday, Jan 28 (23:59 MT))

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**Policy:** You are allowed to reference any material, but please cite the source and write the solutions in your own words. Collaboration is not allowed for this homework.

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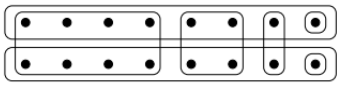
1. (5 points) Recall the local search algorithm for the Max-Cut problem (lecture 1). Construct a graph and a *starting partition* for which the algorithm terminates with a cut of size $\leq (1/2)$ the size of the maximum cut.

2. (5 points) Now consider the greedy algorithm for the Set Cover problem. Construct an instance with $N$ topics, for which the output of the greedy algorithm is $\Omega(\log N)$ factor worse than the optimum.
   *Hint: think of $N$ as a power of 2 and generalize the construction in the following figure (dots are topics..)*



3. (10 points) The *vertex cover* problem is defined as follows. We have a graph $G = (V, E)$, and the goal is to pick the smallest subset $S$ of the vertices, such that for every edge $e \in E$, at least one of its endpoints is in $S$.

   (a) (2 points) Show that the vertex cover problem is a special case of the Set Cover problem.

   (b) (3 points) Consider any *maximal matching* in the graph.[1] Suppose $M$ is the set of edges in this matching. First prove that the set of all end points of the edges in $M$ give a vertex cover. Second, prove that any vertex cover must have size $\geq M$. How does this lead to an approximation algorithm for vertex cover? What is the approximation factor?

   (c) (5 points) Consider the ILP for the vertex cover problem, in which we have variables $x_i$ for the vertices $i$, and for every edge $ij$, we have a constraint $x_i + x_j \geq 1$. Now consider the LP relaxation (so $x_i$ are allowed to be fractional, in the interval $[0, 1]$). What can we say about $\max\{x_i, x_j\}$ for an edge $ij$? Give a rounding algorithm that leads to an approximation factor of 2.

4. (Bonus – not graded) Suppose we pick $n$ random numbers in the interval $[0, 1]$, what is the maximum gap between successive numbers? Can you give upper and lower bounds for this quantity, with high probability?
   *(Hint: divide the interval into equal sized parts, and do a balls and bins style analysis. The lower bound is more tricky..)*

5. (10 points) Markov's inequality states that for a non-negative random variable $X$, for any $t > 0$, we have
$$\Pr[X > t\mathbb{E}[X]] \leq 1/t.$$

[1] Look up the maximum matching problem if you do not know what this means.

(a) (5 points) Give a short proof.

(b) (5 points) Markov's inequality is used typically with $t > 1$, i.e., to say that the likelihood that a random variable is much larger than its expectation is small. Can we also say that a non-negative random variable is not *smaller* than its expectation with reasonable probability? I.e., does the following hold:

$$\Pr[X < (1/10)\mathbb{E}[X]] < 1/1000?$$

If so, prove it. If not, give a counterexample.

6. (10 points) Let us examine the trick we used to write Max-Cut as an ILP. The key was to introduce new variables $x_{uv}$ for every pair of vertices $u, v$, along with the variables $x_u, x_v$, and imposing the following constraints:

$$x_{uv} \in \{0,1\}; \qquad x_{uv} \leq x_u; \qquad x_{uv} \leq x_v; \qquad 1 - x_u - x_v + x_{uv} \geq 0;$$

(a) (5 points) Prove that for any 0/1 values of $x_u$ and $x_v$, the constraints above force $x_{uv}$ to take the value $x_u \cdot x_v$.

(b) (5 points) Suppose we introduce variables $x_{uvw}$ for every *triple* $u, v, w$ of vertices (along with $x_{uv}$'s as above). Can we give linear constraints (along with $x_{uvw} \in \{0,1\}$) that force $x_{uvw}$ to take the value $x_u x_v x_w$?