Opening Thought

Why must functions always have a name?

Anonymous Functions

From now on, functions can be anonymous

- Old code

```
(define (eval-rands rands env fenv)
  (let ([eval-one (lambda (rand)
                         (eval-expression rand env fenv))])
    (map eval-one rands)))
```

- New code

```
(define (eval-rands rands env fenv)
  (map (lambda (rand) (eval-expression rand env fenv))
       rands))
```

Lambda as an Expression

To suport anonymous functions, we must first

- allow (**lambda** (<id>*) <expr>) as an expression
- change the application grammar to (<expr> <expr>*)

<expr> ::= <num>
::= <id>
::= (+ <expr> <expr>)
::= (**let** ([<id> <expr>]*) <expr>)
::= (<expr> <expr>*)
::= (**lambda** (<id>*) <expr>)

<val> ::= <num>
::= (**lambda** (<id>*) <expr>)

Evaluation with Lambda Expressions

Now we need only one kind of **let** form

(let ([identity (lambda (x) x)])
(identity 5))

$\rightarrow$

((**lambda** (**x**) **x**) 5) *usual substitution with values*

$\rightarrow$

5 *new procedure application rule...*

New Application Rule

... ((**lambda** (<id>$_1$...<id>$_k$) <expr>$_a$) <val>$_1$...<val>$_k$) **...**

$\rightarrow$

... <expr>$_b$ **...**

where <expr>$_b$ is <expr>$_a$ with free <id>$_i$ replaced by <val>$_i$

((**lambda** (**x**) **x**) 5) $\rightarrow$ 5

New Application Rule

... ((**lambda** (<id>$_1$...<id>$_k$) <expr>$_a$) <val>$_1$...<val>$_k$) **...**

$\rightarrow$

... <expr>$_b$ **...**

where <expr>$_b$ is <expr>$_a$ with free <id>$_i$ replaced by <val>$_i$

((**lambda** (**x y**) (+ **x y**)) 2 3) $\rightarrow$ (+ 2 3) $\rightarrow$ 5

Using Anonymous Functions

Using anonymous functions, we can easily feed a list of fish:

```
;; feed all fish 1 lb of food:
(map (lambda (x) (+ x 1)) '(4 5 8))
  = '(5 6 9)

;; feed all fish 2 lbs of food:
(map (lambda (x) (+ x 2)) '(5 6 9))
  = '(7 8 11)
```

Using Anonymous Functions

Using anonymous functions, we can easily feed a list of fish:

;; **feed all fish** 1 **lb of food**:
(**map** (**lambda** (**x**) (+ **x** 1)) **'**(4 5 8))
  = **'**(5 6 9)

;; **feed all fish** 2 **lbs of food**:
(**map** (**lambda** (**x**) (+ **x** 2)) **'**(5 6 9))
  = **'**(7 8 11)

Avoid cut-and-paste of the **lambda** expression?

Functions that Return Functions

```
;; make-feeder : <num> → (<num> → <num>)
(define (make-feeder amt)
  (lambda (x) (+ x amt)))

;; feed all fish 1 lb of food:
(map (make-feeder 1) '(4 5 8))
  = '(5 6 9)

;; feed all fish 2 lbs of food:
(map (make-feeder 2) '(5 6 9))
  = '(7 8 11)
```

Another Example with Procedures as Values

(**let** ([**mk-add** (**lambda** (**x**) (**lambda** (**y**) (+ **x** **y**)))])
  (**let** ([**add5** (**mk-add** 5)])
    (**add5** 7)))

$\rightarrow$

(**let** ([**add5** ((**lambda** (**x**) (**lambda** (**y**) (+ **x** **y**))) 5)])
  (**add5** 7))

$\rightarrow$

(**let** ([**add5** (**lambda** (**y**) (+ 5 **y**))])
  (**add5** 7))

$\rightarrow$

((**lambda** (**y**) (+ 5 **y**)) 7)

$\rightarrow$

(+ 5 7) $\rightarrow$ 12

Teminology: First-Order and Higher-Order

- The procedures supported by top-level definitions are ***first-order*** procedures
  - A procedure cannot consume or produce a procedure
  - Methods in Java and procedures in Fortran are first-order
  - Functions C are first-order, but function pointers are values

Teminology: First-Order and Higher-Order

- The procedures supported by **lambda** are ***higher-order*** procedures
  - A procedure can return a procedure that returns a procedure that consumes a procedure that returns a procedure...
  - Procedures in Scheme are higher-order

Procedure Expressions in the Book Language

Concrete extensions:

<prog>	::=	<expr>
<expr>	::=	**proc** (<id>*$^{(,)}$) <expr>
	::=	(<expr> <expr>*)

let identity = **proc**(**x**) **x**
in (**identity** 5)
$\longrightarrow\!\!\!\!\!\longrightarrow$ 5

Procedure Expressions in the Book Language

Concrete extensions:

<prog> ::= <expr>
<expr> ::= **proc** (<id>*^(,)^) <expr>
::= (<expr> <expr>*)

let sum = **proc**(**x**, **y**, **z**) +(**x**, +(**y**, **z**))
in (**sum** 10 20 30)
$\longrightarrow\longrightarrow$ 60

Procedure Expressions in the Book Language

Concrete extensions:

<prog>	::=	<expr>
<expr>	::=	**proc** (<id>*(,)) <expr>
	::=	(<expr> <expr>*)

(**proc**(**x**) **x** 5)

$\rightarrow\rightarrow$ 5

Procedure Expressions in the Book Language

Concrete extensions:

<prog> ::= <expr>
<expr> ::= **proc** (<id>*$^{(,)}$) <expr>
::= (<expr> <expr>*)

let mkadd = **proc**(**x**) **proc**(**y**) +(**x**, **y**)
in let add5 = (**mkadd** 5)
in let x = 10
in (**add5** 6)

$\longrightarrow\longrightarrow$ 11

Evaluation with Environments

Expr	*Env*
let x = +(2, 3) **in x**	{ }

- This trace shows the expression and environment arguments to **`eval-expresson`**

Evaluation with Environments

Expr	*Env*
let x = +(2, 3) **in x**	{ }
➡+(2, 3)	{ }

- Arrows show nested recursive calls

Evaluation with Environments

Expr	*Env*
let x = +(2, 3) **in x**	{ }
➡5	{ }

Evaluation with Environments

Expr	*Env*
let x = +(2, 3) **in x**	{ }
➡5	{ }

- Eventually a value is reached for each recursive call
- To continue with **let**, extend the environment and evaluate the body

Evaluation with Environments

Expr	*Env*
x	{ **x** = 5 }

- Drop the context for the recursive body evaluation, since it isn't needed

Evaluation with Environments

Expr	*Env*
5	{ **x** = 5 }

Evaluation with Environments

Expr	*Env*

let x = 5
  in let x = 6
    in x
	{ }

- Another example: nested **let**

Evaluation with Environments

Expr	*Env*
let x = 5 **in let x** = 6 **in x**	{ }
➡5	{ }

Evaluation with Environments

Expr	*Env*
let x = 5 **in let x** = 6 **in x**	{ }
➡5	{ }

Evaluation with Environments

Expr	*Env*
let x = 6 **in x**	{ **x** = 5 }

Evaluation with Environments

Expr	*Env*
let x = 6 **in x**	{ **x** = 5 }
➡6	{ **x** = 5 }

Evaluation with Environments

Expr	*Env*
let x = 6 **in x**	{ **x** = 5 }
➡6	{ **x** = 5 }

- New value for **x** replaces the old one for the body

Evaluation with Environments

Expr	*Env*
x	{ **x** = 6 }

Evaluation with Environments

Expr	*Env*
6	{ **x** = 6 }

Evaluation with Environments

Expr	*Env*

let x = 5
  in let y = **let x** = 6 **in x**
    in x
	{ }

- Another example: **let** nested in a different way

Evaluation with Environments

Expr	*Env*
let x = 5 **in let y** = **let x** = 6 **in x** **in x**	{ }
➡5	{ }

Evaluation with Environments

Expr	*Env*
let x = 5 **in let y** = **let x** = 6 **in x** **in x**	{ }
➡5	{ }

Evaluation with Environments

Expr	*Env*
let y = **let x** = 6 **in x** **in x**	{ **x** = 5 }

Evaluation with Environments

Expr	*Env*
let y = **let x** = 6 **in x** **in x**	{ **x** = 5 }
➡**let x** = 6 **in x**	{ **x** = 5 }

Evaluation with Environments

Expr	*Env*
let y = **let x** = 6 **in x** **in x**	{ **x** = 5 }
➡**let x** = 6 **in x**	{ **x** = 5 }
➡6	{ **x** = 5 }

Evaluation with Environments

Expr	*Env*
let y = **let x** = 6 **in x** **in x**	{ **x** = 5 }
➡**let x** = 6 **in x**	{ **x** = 5 }
➡6	{ **x** = 5 }

Evaluation with Environments

Expr	*Env*
let y = **let x** = 6 **in x** **in x**	{ **x** = 5 }
➡**x**	{ **x** = 6 }

Evaluation with Environments

Expr	*Env*
let y = **let x** = 6 **in x** **in x**	{ **x** = 5 }
➡6	{ **x** = 6 }

Evaluation with Environments

Expr	*Env*
let y = **let x** = 6 **in x** **in x**	{ **x** = 5 }
➡6	{ **x** = 6 }

- What environment is extended with **y** = 6?

Evaluation with Environments

Expr	*Env*
let y = **let x** = 6 **in x** **in x**	{ **x** = 5 }
➡6	{ **x** = 6 }

- Answer: the original one for the **let** of **y**

Evaluation with Environments

Expr	*Env*
x	{ **x** = 5, **y** = 6 }

Evaluation with Environments

Expr	*Env*
5	{ **x** = 5, **y** = 6 }

Evaluation with Procedures and Environments

Expr	*Env*
let mkadd = **proc**(**x**) **proc**(**y**) +(**x**, **y**) **in let add5** = (**mkadd** 5) **in** (**add5** 6)	{ }

Evaluation with Procedures and Environments

Expr	*Env*
let mkadd = **proc**(**x**) **proc**(**y**) +(**x**, **y**) **in let add5** = (**mkadd** 5) **in** (**add5** 6)	{ }
➡**proc**(**x**) **proc**(**y**) +(**x**, **y**)	{}

- Is a **proc** expression a value?
- A **lambda** was a value in Scheme... so let's say it's ok

this choice will turn out to be slightly wrong

Evaluation with Procedures and Environments

Expr	*Env*
let mkadd = **proc**(**x**) **proc**(**y**) +(**x**, **y**) **in let add5** = (**mkadd** 5) **in** (**add5** 6)	{ }
➡**proc**(**x**) **proc**(**y**) +(**x**, **y**)	{}

Evaluation with Procedures and Environments

Expr	*Env*
let add5 = (**mkadd** 5) **in** (**add5** 6)	{ **mkadd** = **proc**(**x**) **proc**(**y**) +(**x**, **y**) }

Evaluation with Procedures and Environments

Expr	*Env*
let add5 = (**mkadd** 5) **in** (**add5** 6)	{ **mkadd** = **proc**(**x**) **proc**(**y**) +(**x**, **y**) }
➡(**mkadd** 5)	{ **mkadd** = **proc**(**x**) **proc**(**y**) +(**x**, **y**) }

Evaluation with Procedures and Environments

Expr	*Env*
let add5 = (**mkadd** 5) **in** (**add5** 6)	{ **mkadd** = **proc**(**x**) **proc**(**y**) +(**x**, **y**) }
➡(**mkadd** 5)	{ **mkadd** = **proc**(**x**) **proc**(**y**) +(**x**, **y**) }
➡**mkadd**	{ **mkadd** = **proc**(**x**) **proc**(**y**) +(**x**, **y**) }

Evaluation with Procedures and Environments

Expr	*Env*
let add5 = (**mkadd** 5) **in** (**add5** 6)	{ **mkadd** = **proc**(**x**) **proc**(**y**) +(**x**, **y**) }
➡(**mkadd** 5)	{ **mkadd** = **proc**(**x**) **proc**(**y**) +(**x**, **y**) }
➡**proc**(**x**) **proc**(**y**) +(**x**, **y**)	{ **mkadd** = **proc**(**x**) **proc**(**y**) +(**x**, **y**) }

Evaluation with Procedures and Environments

Expr	*Env*
let add5 = (**mkadd** 5) **in** (**add5** 6)	{ **mkadd** = **proc**(**x**) **proc**(**y**) +(**x**, **y**) }
(**mkadd** 5)	{ **mkadd** = **proc**(**x**) **proc**(**y**) +(**x**, **y**) }
proc(**x**) **proc**(**y**) +(**x**, **y**)	{ **mkadd** = **proc**(**x**) **proc**(**y**) +(**x**, **y**) }
5	{ **mkadd** = **proc**(**x**) **proc**(**y**) +(**x**, **y**) }

Evaluation with Procedures and Environments

Expr	*Env*
let add5 = (**mkadd** 5) **in** (**add5** 6)	{ **mkadd** = **proc**(**x**) **proc**(**y**) +(**x**, **y**) }
➡(**mkadd** 5)	{ **mkadd** = **proc**(**x**) **proc**(**y**) +(**x**, **y**) }
➡**proc**(**x**) **proc**(**y**) +(**x**, **y**)	{ **mkadd** = **proc**(**x**) **proc**(**y**) +(**x**, **y**) }
➡5	{ **mkadd** = **proc**(**x**) **proc**(**y**) +(**x**, **y**) }

- To evaluate an application, extend the application's environment with a binding for the argument

this isn't quite right, either

Evaluation with Procedures and Environments

Expr	*Env*
let add5 = (**mkadd** 5) **in** (**add5** 6)	{ **mkadd** = **proc**(**x**) **proc**(**y**) +(**x**, **y**) }
proc (**y**) +(**x**, **y**)	{ **mkadd** = **proc**(**x**) **proc**(**y**) +(**x**, **y**) **x** = 5 }

Evaluation with Procedures and Environments

Expr	*Env*
let add5 = (**mkadd** 5) **in** (**add5** 6)	{ **mkadd** = **proc**(**x**) **proc**(**y**) +(**x**, **y**) }
➡**proc** (**y**) +(**x**, **y**)	{ **mkadd** = **proc**(**x**) **proc**(**y**) +(**x**, **y**) **x** = 5 }

- So the value for **add5** is also a procedure
- Extend the original environment for the **let**

Evaluation with Procedures and Environments

Expr	*Env*
(**add5** 6)	{ **mkadd** = **proc**(**x**) **proc**(**y**) +(**x**, **y**) **add5** = **proc** (**y**) +(**x**, **y**) }

- We can see where this is going... **x** has no value
- What went wrong?

Evaluation with Procedures and Environments

Expr	*Env*
(**add5** 6)	{ **mkadd** = **proc**(**x**) **proc**(**y**) +(**x**, **y**) **add5** = **proc** (**y**) +(**x**, **y**) }

- In Scheme, procedures as values worked because they had eager substitutions

Evaluation with Procedures and Environments

Expr	*Env*
(**add5** 6)	{ **mkadd** = **proc**(**x**) **proc**(**y**) +(**x**, **y**) **add5** = **proc** (**y**) +(**x**, **y**) }

- With lazy substitutions: combine a **proc** and an environment to get a value
- The combination is called a ***closure***

Evaluation with Closures

Expr	*Env*
let mkadd = **proc**(**x**) **proc**(**y**) +(**x**, **y**)	
in let add5 = (**mkadd** 5)	{ }
in (**add5** 6)	

Evaluation with Closures

Expr	*Env*
let mkadd = **proc**(**x**) **proc**(**y**) +(**x**, **y**) **in let add5** = (**mkadd** 5) **in** (**add5** 6)	{ }
➡**proc**(**x**) **proc**(**y**) +(**x**, **y**)	{}

Evaluation with Closures

Expr	*Env*
let mkadd = **proc**(**x**) **proc**(**y**) +(**x**, **y**) **in let add5** = (**mkadd** 5) **in** (**add5** 6)	{ }
<**proc**(**x**) **proc**(**y**) +(**x**, **y**), { }>	{}

- Create a closure with the current environment to get a value

Evaluation with Closures

Expr	*Env*
let mkadd = **proc**(**x**) **proc**(**y**) +(**x**, **y**) **in let add5** = (**mkadd** 5) **in** (**add5** 6)	{ }
<(**x**), **proc**(**y**) +(**x**, **y**), { }>	{}

- Alternate form: arguments, body, and environment

Evaluation with Closures

Expr	*Env*
let mkadd = **proc**(**x**) **proc**(**y**) +(**x**, **y**) **in let add5** = (**mkadd** 5) **in** (**add5** 6)	{ }
➡<(**x**), **proc**(**y**) +(**x**, **y**), { }>	{}

- A closure is a value

Evaluation with Closures

Expr	*Env*
let add5 = (**mkadd** 5) **in** (**add5** 6)	{ **mkadd** = <(**x**), **proc**(**y**) +(**x**, **y**), { }> }

Evaluation with Closures

Expr	*Env*
let add5 = (**mkadd** 5) **in** (**add5** 6)	{ **mkadd** = <(**x**), **proc**(**y**) +(**x**, **y**), { }> }
➡(**mkadd** 5)	{ **mkadd** = <(**x**), **proc**(**y**) +(**x**, **y**), { }> }

Evaluation with Closures

Expr	*Env*
let add5 = (**mkadd** 5) **in** (**add5** 6)	{ **mkadd** = <(**x**), **proc**(**y**) +(**x**, **y**), { }> }
➡(**mkadd** 5)	{ **mkadd** = <(**x**), **proc**(**y**) +(**x**, **y**), { }> }
➡**mkadd**	{ **mkadd** = <(**x**), **proc**(**y**) +(**x**, **y**), { }> }

Evaluation with Closures

Expr	*Env*
let add5 = (**mkadd** 5) **in** (**add5** 6)	{ **mkadd** = <(**x**), **proc**(**y**) +(**x**, **y**), { }> }
➡(**mkadd** 5)	{ **mkadd** = <(**x**), **proc**(**y**) +(**x**, **y**), { }> }
➡<(**x**), **proc**(**y**) +(**x**, **y**), { }>	{ **mkadd** = <(**x**), **proc**(**y**) +(**x**, **y**), { }> }

Evaluation with Closures

Expr	*Env*
let add5 = (**mkadd** 5) **in** (**add5** 6)	{ **mkadd** = <(**x**), **proc**(**y**) +(**x**, **y**), { }> }
(**mkadd** 5)	{ **mkadd** = <(**x**), **proc**(**y**) +(**x**, **y**), { }> }
<(**x**), **proc**(**y**) +(**x**, **y**), { }>	{ **mkadd** = <(**x**), **proc**(**y**) +(**x**, **y**), { }> }
5	{ **mkadd** = <(**x**), **proc**(**y**) +(**x**, **y**), { }> }

Evaluation with Closures

Expr	*Env*
let add5 = (**mkadd** 5) **in** (**add5** 6)	{ **mkadd** = <(**x**), **proc**(**y**) +(**x**, **y**), { }> }
→(**mkadd** 5)	{ **mkadd** = <(**x**), **proc**(**y**) +(**x**, **y**), { }> }
→<(**x**), **proc**(**y**) +(**x**, **y**), { }>	{ **mkadd** = <(**x**), **proc**(**y**) +(**x**, **y**), { }> }
→5	{ **mkadd** = <(**x**), **proc**(**y**) +(**x**, **y**), { }> }

- To evaluate an application, extend the *closure's* environment with a binding for the argument

Evaluation with Closures

Expr	*Env*
let add5 = (**mkadd** 5) **in** (**add5** 6)	{ **mkadd** = <(**x**), **proc**(**y**) +(**x**, **y**), { }> }
proc (**y**) +(**x**, **y**)	{ **x** = 5 }

Evaluation with Closures

Expr	*Env*
let add5 = (**mkadd** 5) **in** (**add5** 6)	{ **mkadd** = <(**x**), **proc**(**y**) +(**x**, **y**), { }> }
<(**y**), +(**x**, **y**), { **x** = 5 }>	{ **x** = 5 }

- Again, create a closure
- Note that the **x** binding is saved in the closure

Evaluation with Closures

Expr	*Env*
let add5 = (**mkadd** 5) **in** (**add5** 6)	{ **mkadd** = <(**x**), **proc**(**y**) +(**x**, **y**), { }> }
<(**y**), +(**x**, **y**), { **x** = 5 }>	{ **x** = 5 }

Evaluation with Closures

Expr	*Env*
(**add5** 6)	{ **mkadd** = <(**x**), **proc**(**y**) +(**x**, **y**), { }> **add5** = <(**y**), +(**x**, **y**), { **x** = 5 }> }

Evaluation with Closures

Expr	*Env*
(**add5** 6)	{ **mkadd** = <(**x**), **proc**(**y**) +(**x**, **y**), { }> **add5** = <(**y**), +(**x**, **y**), { **x** = 5 }> }
add5	{ **mkadd** = <(**x**), **proc**(**y**) +(**x**, **y**), { }> **add5** = <(**y**), +(**x**, **y**), { **x** = 5 }> }

Evaluation with Closures

Expr	*Env*
(**add5** 6)	{ **mkadd** = <(**x**), **proc**(**y**) +(**x**, **y**), { }> **add5** = <(**y**), +(**x**, **y**), { **x** = 5 }> }
➡<(**y**), +(**x**, **y**), { **x** = 5 }>	{ **mkadd** = <(**x**), **proc**(**y**) +(**x**, **y**), { }> **add5** = <(**y**), +(**x**, **y**), { **x** = 5 }> }

Evaluation with Closures

Expr	*Env*
(**add5** 6)	{ **mkadd** = <(**x**), **proc**(**y**) +(**x**, **y**), { }> **add5** = <(**y**), +(**x**, **y**), { **x** = 5 }> }
<(**y**), +(**x**, **y**), { **x** = 5 }>	{ **mkadd** = <(**x**), **proc**(**y**) +(**x**, **y**), { }> **add5** = <(**y**), +(**x**, **y**), { **x** = 5 }> }
6	{ **mkadd** = <(**x**), **proc**(**y**) +(**x**, **y**), { }> **add5** = <(**y**), +(**x**, **y**), { **x** = 5 }> }

Evaluation with Closures

Expr	*Env*
(**add5** 6)	{ **mkadd** = <(**x**), **proc**(**y**) +(**x**, **y**), { }> **add5** = <(**y**), +(**x**, **y**), { **x** = 5 }> }
→ <(**y**), +(**x**, **y**), { **x** = 5 }>	{ **mkadd** = <(**x**), **proc**(**y**) +(**x**, **y**), { }> **add5** = <(**y**), +(**x**, **y**), { **x** = 5 }> }
→ 6	{ **mkadd** = <(**x**), **proc**(**y**) +(**x**, **y**), { }> **add5** = <(**y**), +(**x**, **y**), { **x** = 5 }> }

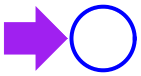
- Extend the closure's environment { **x** = 5 } with a binding for **y**

Evaluation with Closures

Expr	*Env*
+(**x**, **y**)	{ **x** = 5, **y** = 6 }

- This is clearly going to work

Environments in Picture Form

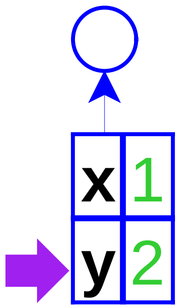


top purple arrow points to the current environment

purple in bottom area hilites the current expression

let x = 1 **y** = 2
  in +(**x**, **y**)

Environments in Picture Form

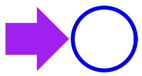


top purple arrow points to the current environment

purple in bottom area hilites the current expression

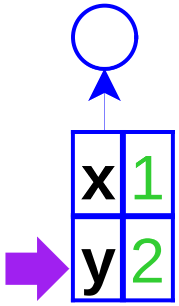
let x = 1 **y** = 2
  in **+(x, y)**

Environments in Picture Form



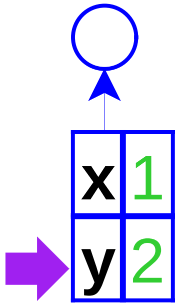
```
let x = 1  y = 2
  in let f = proc (z) +(z, y)
    in (f y)
```

Environments in Picture Form



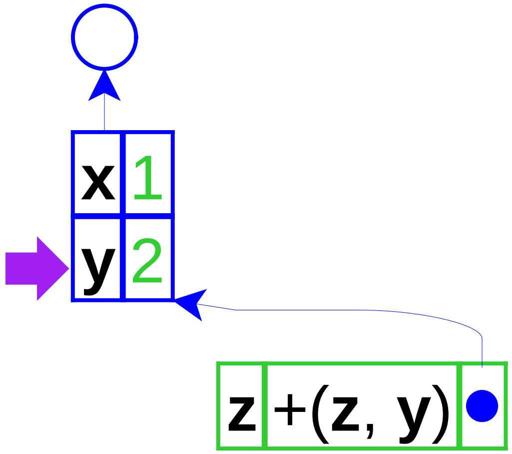
let x = 1 **y** = 2
in let f = proc (z) +(z, y)
in (f y)

Environments in Picture Form



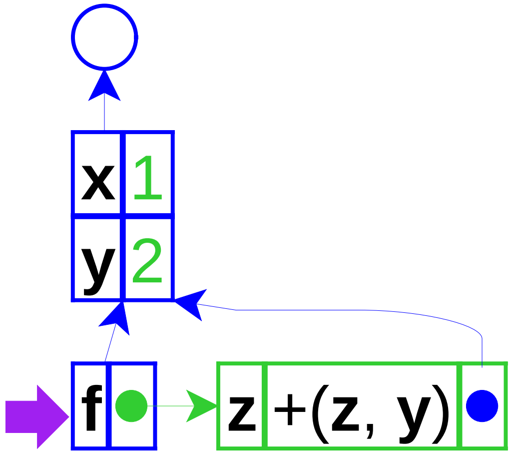
let x = 1 **y** = 2
  in let f = **proc (z) +(z, y)**
    in (**f y**)

Environments in Picture Form



let x = 1 **y** = 2
  in let f = **proc (z) +(z, y)**
    in (f y)

Environments in Picture Form

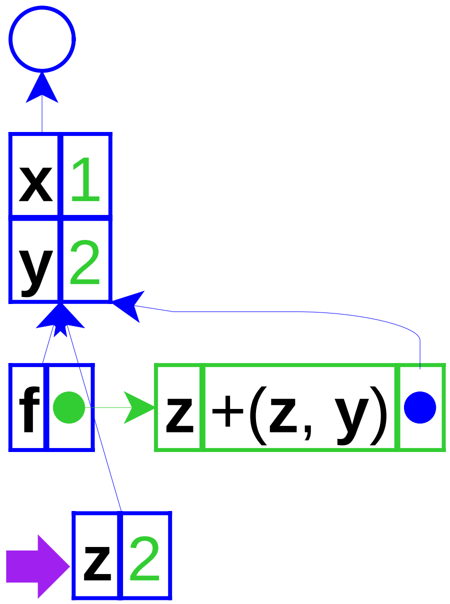


let x = 1 **y** = 2

  in let f = **proc** (**z**) +(**z**, **y**)

    in (f y)

Environments in Picture Form



let x = 1 **y** = 2
  in let f = **proc** (**z**) +(**z**, **y**)
    in (**f y**)

Procedure Expressions in the Book Language

Abstract extensions:

<prog>	::=	(**a-program** <expr>)
<expr>	::=	(**proc-exp** (**list** <id>*) <expr>)
	::=	(**app-exp** <expr> (**list** <expr>*))
<val>	::=	<num>
	::=	<proc>
<proc>	::=	(**closure** (**list** <id>*) <expr> <env>)