Typing Example: Number

$$\{\} \vdash 5 : \texttt{int}$$

Each

$$E \vdash e : T$$

is a call to **`type-of-expression`** with arguments e and E where the result is T

Typing Example: Sum

$$\frac{\{\} \vdash 1 : \texttt{int} \qquad \{\} \vdash 2 : \texttt{int}}{\{\} \vdash +(1,2) : \texttt{int}}$$

- Actually, the type checker treats primitives like functions, but it could be checked directly as above
- The above strategy is a good one for HW7, because primitive checking is different than function checking

Typing Example: Function

$$\dfrac{\dfrac{\{\,\mathbf{x} : \texttt{int}\,\} \vdash \mathbf{x} : \texttt{int} \qquad \{\,\mathbf{x} : \texttt{int}\,\} \vdash 2 : \texttt{int}}{\{\,\mathbf{x} : \texttt{int}\,\} \vdash +(\mathbf{x},2) : \texttt{int}}}{\{\,\} \vdash \mathbf{proc}(\mathbf{int}\ \mathbf{x})\ +(\mathbf{x},2) : (\texttt{int} \to \texttt{int})}$$

Typing Example: Function Call

$$\dfrac{\dfrac{\{\,\mathbf{x} : \texttt{int}\,\} \vdash \mathbf{x} : \texttt{int}}{\{\,\} \vdash \mathbf{proc(int\ x)x} : (\texttt{int} \rightarrow \texttt{int})} \qquad \{\,\} \vdash 12 : \texttt{int}}{\{\,\} \vdash (\mathbf{proc(int\ x)x}\ \ 12) : \mathbf{T_2}}$$

$$(\texttt{int} \rightarrow \texttt{int}) = (\texttt{int} \rightarrow \mathbf{T_2})$$

simplified: `int`

- For inference, create a new type variable for each application

Typing Example: ? Argument

$$\dfrac{\dfrac{\{\,\mathbf{x} : \mathbf{T_1}\,\} \vdash \mathbf{x} : \mathbf{T_1} \qquad \{\,\mathbf{x} : \mathbf{T_1}\,\} \vdash 2 : \texttt{int}}{\{\,\mathbf{x} : \mathbf{T_1}\,\} \vdash +(\mathbf{x},2) : \texttt{int}}}{\{\,\} \vdash \mathbf{proc}(?\ \mathbf{x})\ +(\mathbf{x},2) : (\mathbf{T_1} \rightarrow \texttt{int})}$$

$$\mathbf{T_1} = \texttt{int}$$

simplified: $(\texttt{int} \rightarrow \texttt{int})$

- Create a new type variable for each ?

Typing Example: ? Argument

$$\dfrac{\dfrac{\{\,\mathbf{x} : \mathbf{T}_1\,\} \vdash \mathbf{x} : \mathbf{T}_1 \qquad \{\,\mathbf{x} : \mathbf{T}_1\,\} \vdash 2 : \texttt{int} \qquad \{\,\mathbf{x} : \mathbf{T}_1\,\} \vdash 3 : \texttt{int}}{\{\,\mathbf{x} : \mathbf{T}_1\,\} \vdash \textbf{if x then } 2 \textbf{ else } 3 : \texttt{int}}}{\{\,\} \vdash \textbf{proc}(?\ \mathbf{x})\ \textbf{if x then } 2 \textbf{ else } 3 : (\mathbf{T}_1 \rightarrow \texttt{int})}$$

$$\mathbf{T}_1 = \texttt{bool}$$

simplified: $(\texttt{bool} \rightarrow \texttt{int})$

Typing Example: Function-Calling Function

$$\dfrac{\dfrac{\{\,\mathbf{f} : \mathbf{T}_1\,\} \vdash \mathbf{f} : \mathbf{T}_1 \qquad \{\,\mathbf{f} : \mathbf{T}_1\,\} \vdash 12 : \texttt{int}}{\{\,\mathbf{f} : \mathbf{T}_1\,\} \vdash (\mathbf{f}\ 12) : \mathbf{T}_2}}{\{\,\} \vdash \mathbf{proc}(?\ \mathbf{f})(\mathbf{f}\ 12) : (\mathbf{T}_1 \rightarrow \mathbf{T}_2)}$$

$$\mathbf{T}_1 = (\texttt{int} \rightarrow \mathbf{T}_2)$$

simplified: $((\texttt{int} \rightarrow \mathbf{T}_2) \rightarrow \mathbf{T}_2)$

Typing Example: Identity

$$\frac{\{\, \mathbf{x} : \mathbf{T_1} \,\} \vdash \mathbf{x} : \mathbf{T_1}}{\{\,\} \vdash \mathbf{proc}(?\ \mathbf{x})\ \mathbf{x} : (\mathbf{T_1} \rightarrow \mathbf{T_1})}$$

no simplification possible

Typing Example: Identity Applied

$$\dfrac{\dfrac{\{\,\mathbf{x} : \mathbf{T}_1\,\} \vdash \mathbf{x} : \mathbf{T}_1}{\{\,\} \vdash \mathbf{proc}(?\ \mathbf{x})\ \mathbf{x} : (\mathbf{T}_1 \rightarrow \mathbf{T}_1)} \qquad \{\,\} \vdash \mathbf{false} : \mathtt{bool}}{\{\,\} \vdash (\mathbf{proc}(?\ \mathbf{x})\mathbf{x}\ \ \mathbf{false}) : \mathbf{T}_2}$$

$$(\mathbf{T}_1 \rightarrow \mathbf{T}_1) = (\mathtt{bool} \rightarrow \mathbf{T}_2)$$

simplfied: `bool`

Typing Example: Function-Making Function

$$\dfrac{\dfrac{\{\,\mathbf{x} : \mathbf{T}_1,\ \mathbf{y} : \mathbf{T}_2\,\} \vdash \mathbf{x} : \mathbf{T}_1}{\{\,\mathbf{x} : \mathbf{T}_1\,\} \vdash \mathbf{proc}(?\ \mathbf{y})\ \mathbf{x} : (\mathbf{T}_2 \to \mathbf{T}_1)}}{\{\,\} \vdash \mathbf{proc}(?\ \mathbf{x})\ \mathbf{proc}(?\ \mathbf{y})\ \mathbf{x} : (\mathbf{T}_1 \to (\mathbf{T}_2 \to \mathbf{T}_1))}$$

no simplification possible

Typing Example: Compound Primitive Data

$$\frac{\{\} \vdash 1 : \texttt{int} \qquad \{\} \vdash 2 : \texttt{int}}{\{\} \vdash \textbf{cons}(1,2) : [\texttt{int} : \texttt{int}]}$$

- In general, $[\mathbf{T}_1 : \mathbf{T}_2]$ means a pair whose first element is of type $\mathbf{T}_1$ and second element is of type $\mathbf{T}_2$
- More conventional notation is $(\mathbf{T}_1 \times \mathbf{T}_2)$

Typing Example: Compound Primitive Data

$$\frac{\{\} \vdash 1 : \texttt{int} \qquad \{\} \vdash 2 : \texttt{int}}{\{\} \vdash \textbf{cons}(1,2) : [\texttt{int} : \texttt{int}]}$$

General rule:

$$\frac{E \vdash e_1 : T_1 \qquad E \vdash e_2 : T_2}{E \vdash \textbf{cons}(e_1, e_2) : [T_1 : T_2]}$$

Typing Example: Compound Primitive Data

$$\frac{\{\} \vdash \textbf{cons}(1,2) : \texttt{[int : int]}}{\{\} \vdash \textbf{car}(\textbf{cons}(1,2)) : \texttt{int}}$$

General rule:

$$\frac{E \vdash e : [T_1 : T_2]}{E \vdash \textbf{car}(e) : T_1}$$

$$\frac{E \vdash e : [T_1 : T_2]}{E \vdash \textbf{cdr}(e) : T_2}$$

Infinite Loops

What if we extend the language with a special , expression that loops forever?

- **if true then** 1 **else** , $\rightarrow\rightarrow$ 1
- **if false then** 1 **else** , $\rightarrow\rightarrow$ *loops forever*
- **if true then proc**(? **x**)**x else** , $\rightarrow\rightarrow$ **proc**(? **x**)**x**

What is the type of ,?

For HW7, it's `int`, but more generally...

Typing Example: Infinite Loop

$$\frac{\{\} \vdash \textbf{true} : \texttt{bool} \qquad \{\} \vdash 1 : \texttt{int} \qquad \{\} \vdash \ \texttt{,} : \mathbf{T_1}}{\{\} \vdash \textbf{if true then } 1 \textbf{ else } \texttt{,} : \texttt{int}}$$

$$\mathbf{T_1} = \texttt{int}$$

- Create a new type variable for each `,`

Type Inference Summary

- New type variable for each ?
- New type variable for each application
- New type variable for each `,`
- Checking a type equation can force a type variable to match a certain type

The Universe of Programs

- The goal of type-checking is to rule out bad programs

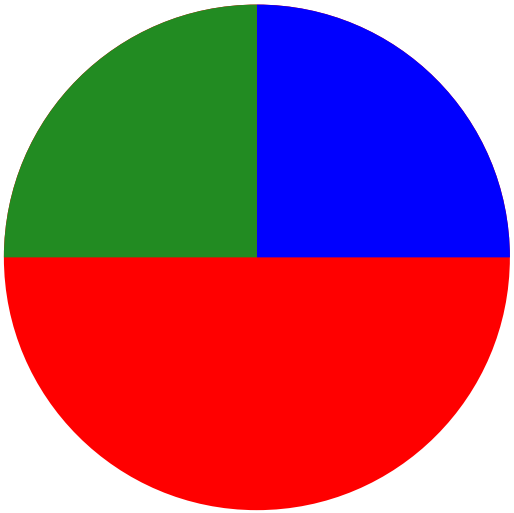
+(1, **true**)

- Unfortunately, some good programs will be ruled out, too

+(1, **if true then** 1 **else false**)

The Universe of Programs

programs that run forever

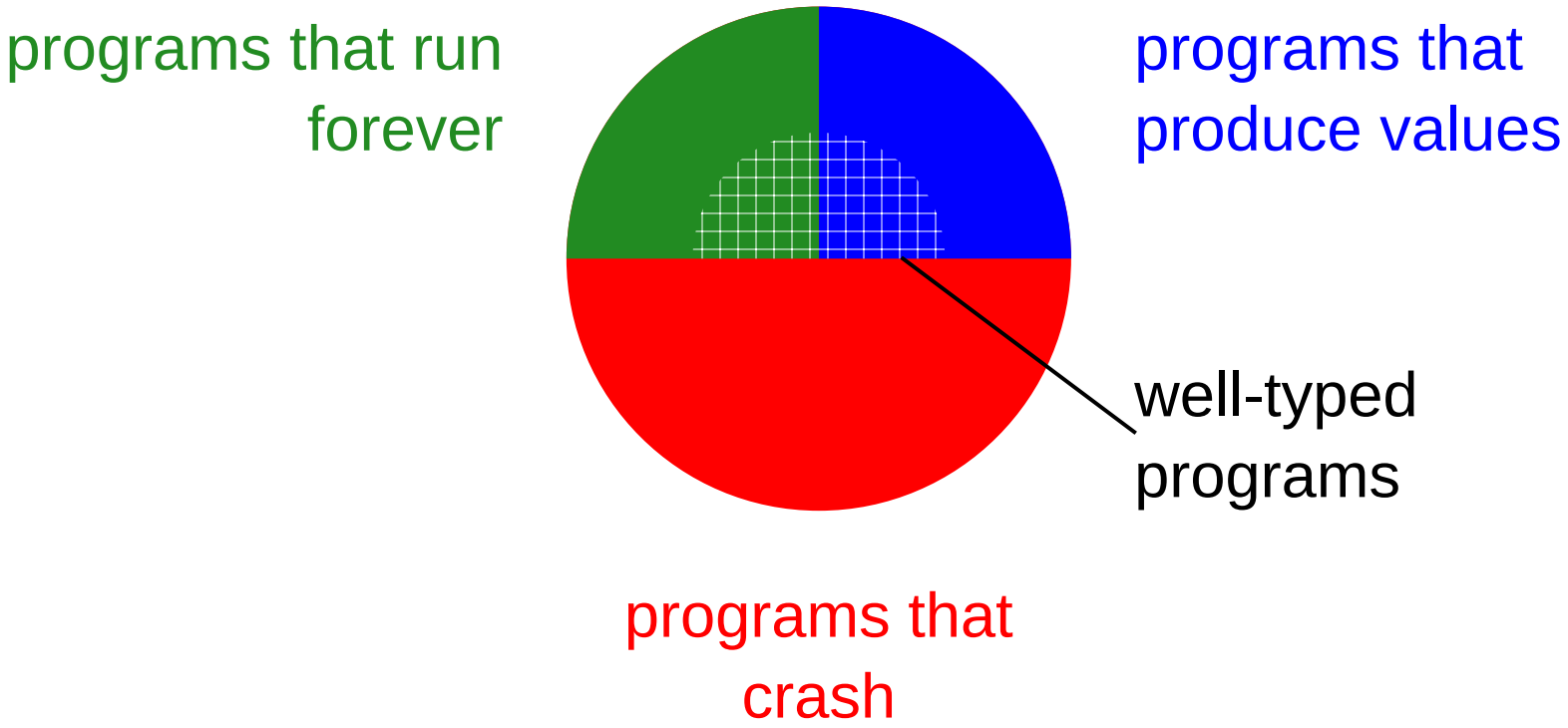


programs that produce values

programs that crash

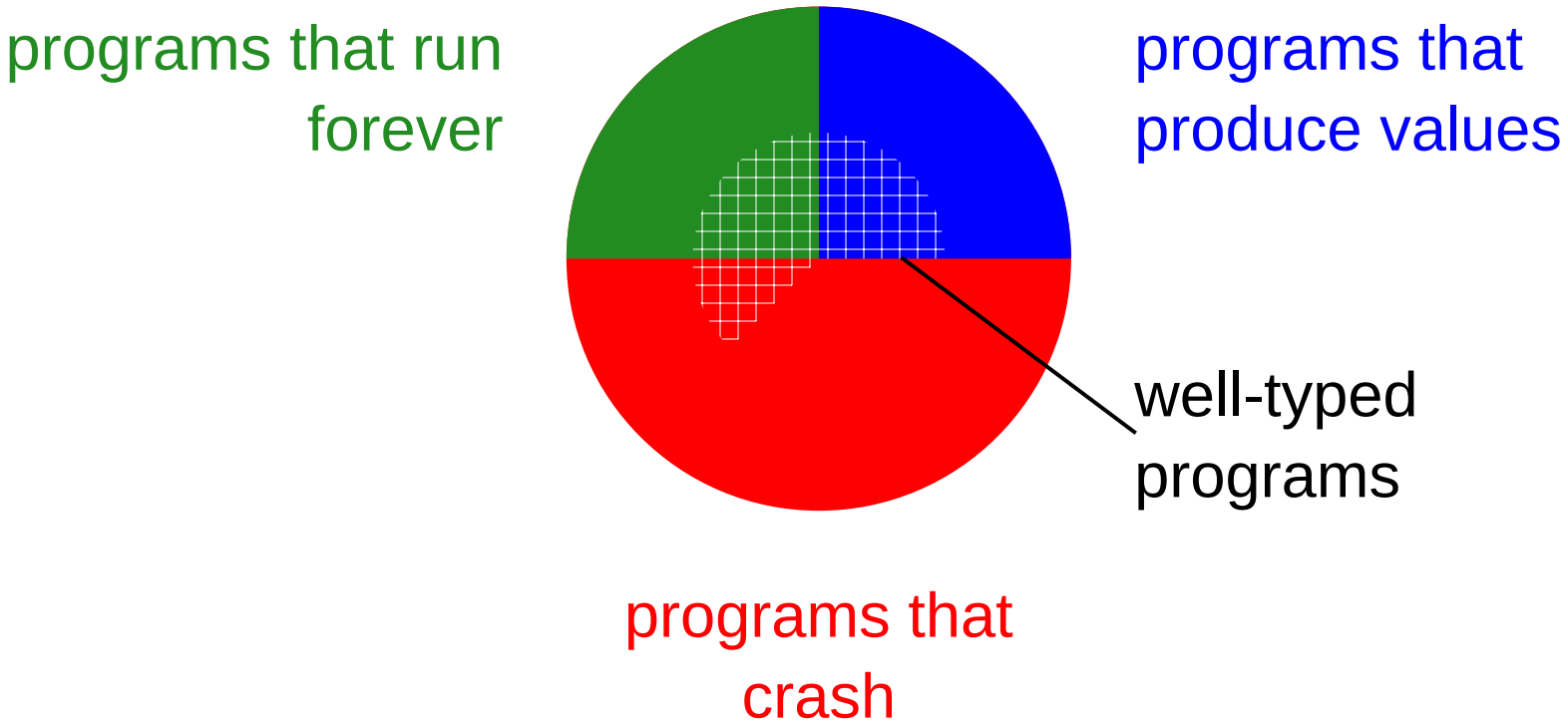
- Every program falls into one of three categories

The Universe of Programs



- The idea is that a type checker rules out the error category

The Universe of Programs



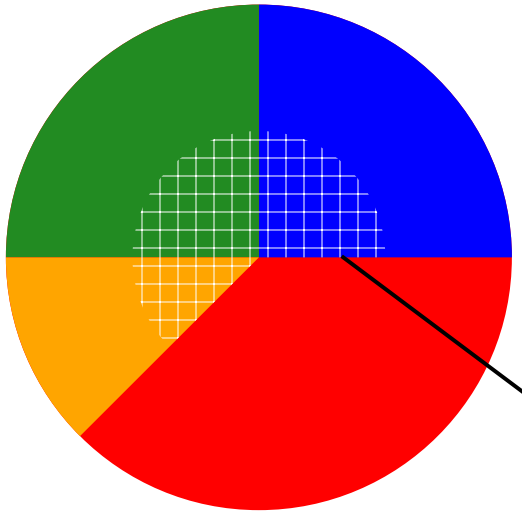
- But a type checker for most languages will allow some errors!

1 / 0 $\rightarrow\rightarrow$ **divide by zero**

The Universe of Programs

programs that run forever

programs that produce values



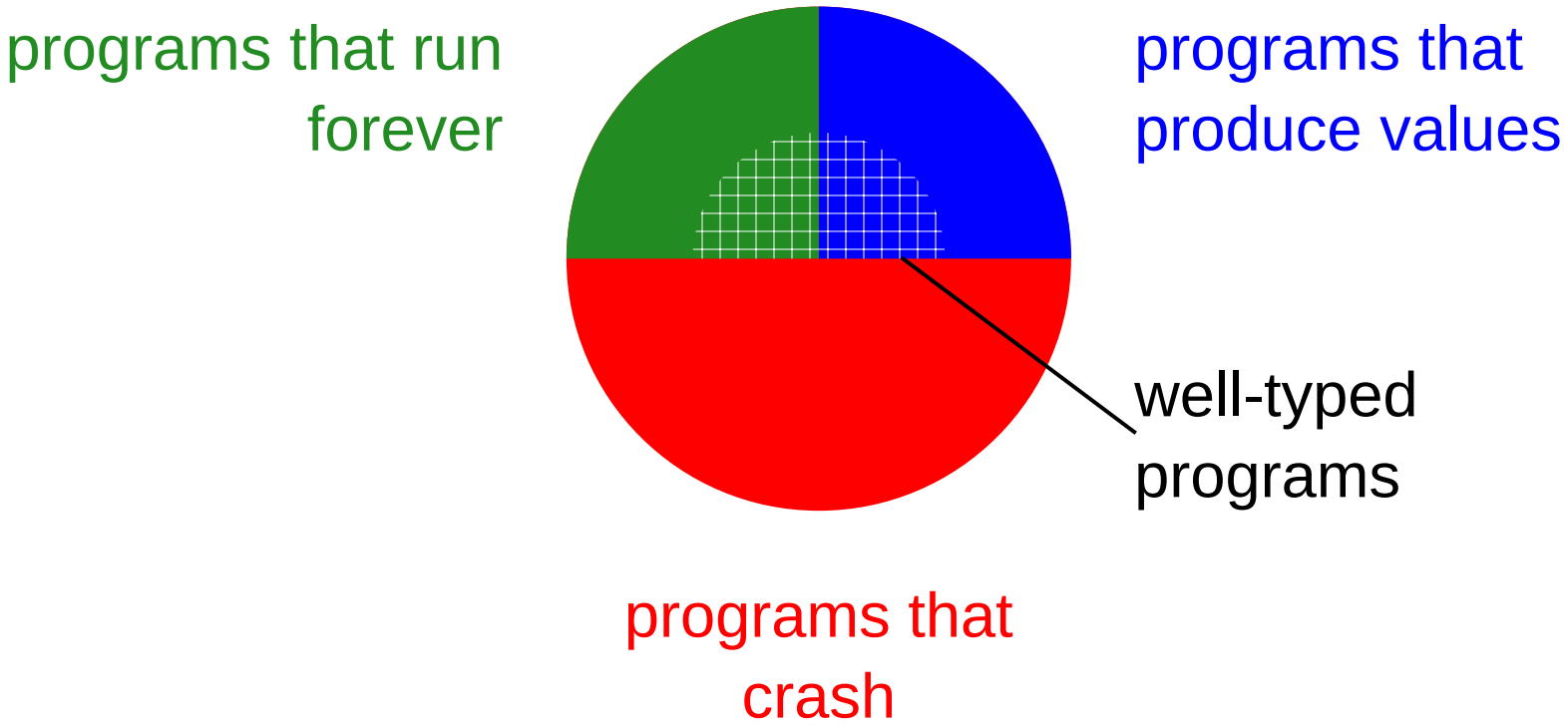
programs that crash on **variants**

well-typed programs

programs that crash on **types**

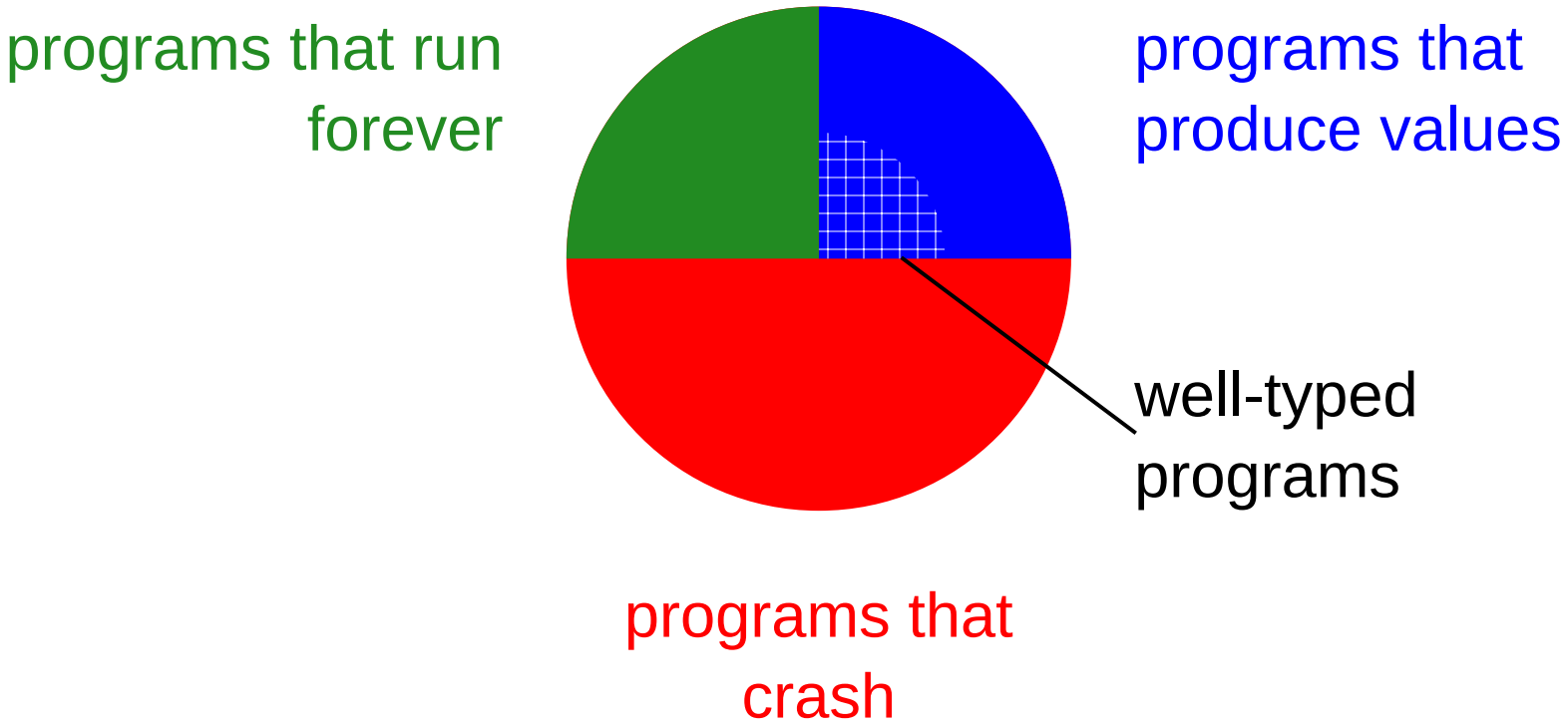
- Still, a type checker *always* rules out a certain class of errors
  - Division by 0 is a ***variant error***

The Universe of Programs



- Our language happens to have no variant errors, so the type checker rules out all errors

The Universe of Programs



- In fact, if we get rid of **letrec**, then every well-typed program terminates with a value!

Intution for Termination

Recall that to get rid of **letrec**

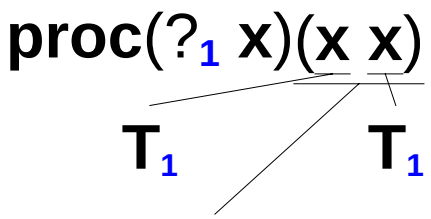
```
letrec int sum = proc(int x)
                   if zero?(x)
                     then 0
                     else +(x,(sum -(x, 1)))
  in (sum 10)
```

we can use self-application:

```
let sum = proc(int x, ? sum)
            if zero?(x)
              then 0
              else +(x,((sum sum) -(x, 1)))
  in ((sum sum) 10)
```

Intution for Termination

But we've already seen that we can't type self-application:



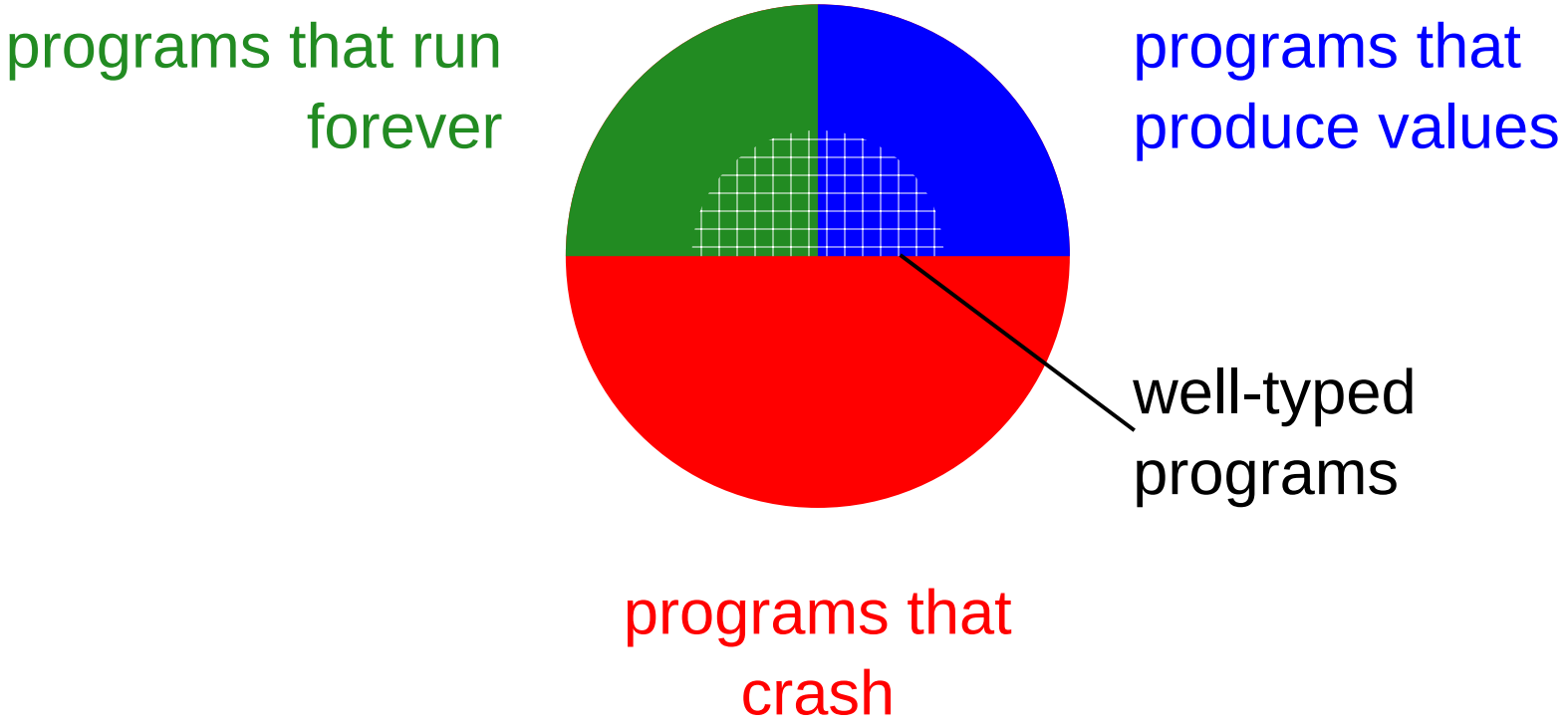
no type: $\mathbf{T_1}$ can't be $(\mathbf{T_1} \rightarrow \mathbf{T_2})$

The only way around this restriction is to restore **letrec** or extend the type language.

(Extending the type language in this direction is beyond the scope of the course.)

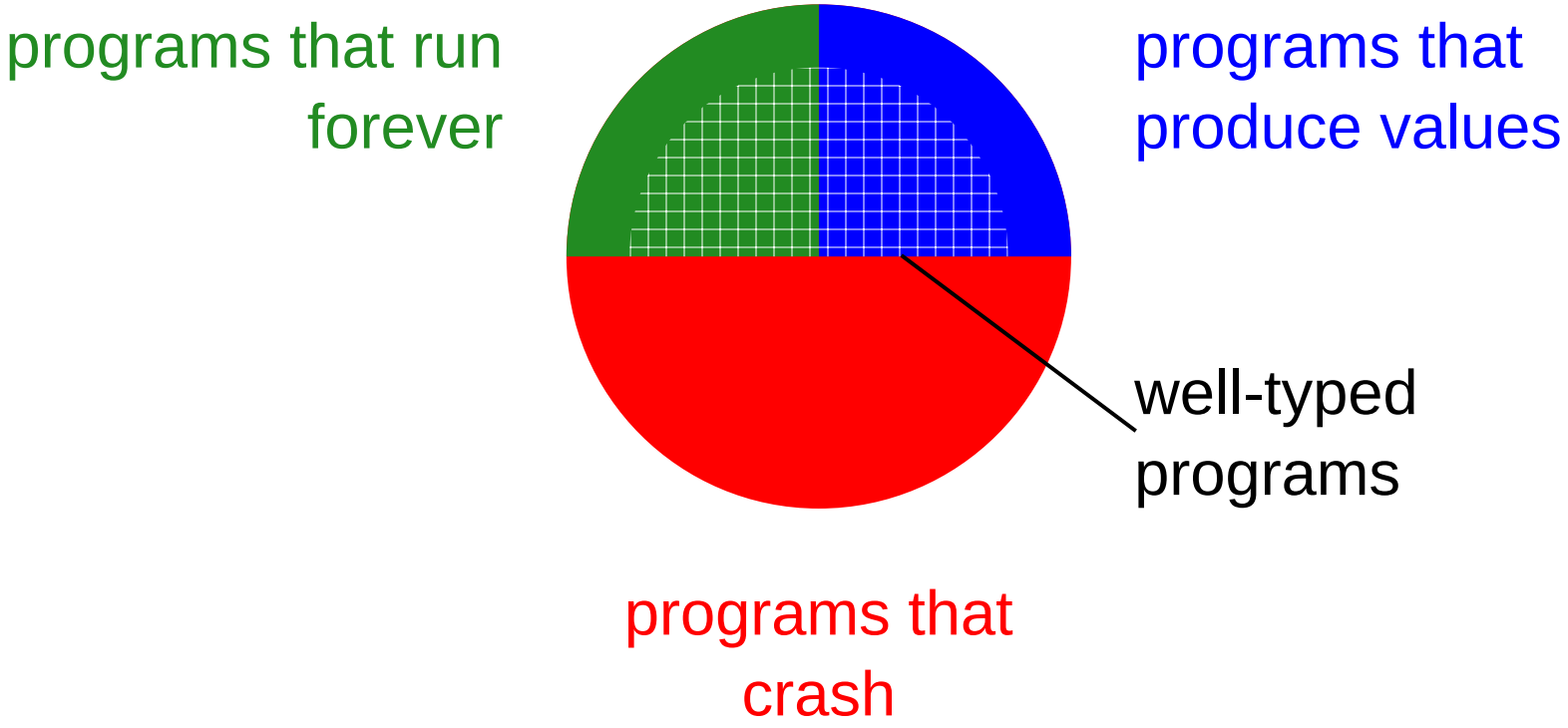
The Universe of Programs

- There are other ways that we'd like to expand the set of well-formed programs



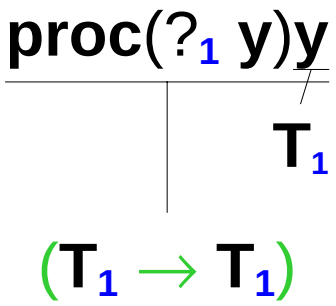
The Universe of Programs

- There are other ways that we'd like to expand the set of well-formed programs



- Adjusting the type rules can allow more programs

Polymorphism



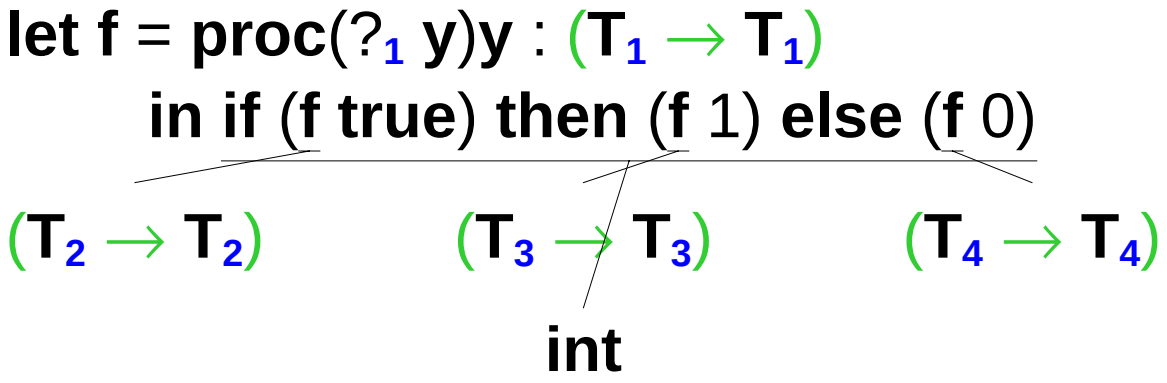
let f = proc(?$_1$ y)y : ($T_1 \rightarrow T_1$)
in if (f true) then (f 1**) else (f** 0**)**

($T_1 \rightarrow T_1$) ($T_1 \rightarrow T_1$) ($T_1 \rightarrow T_1$)

no type: T_1 can't be both `bool` and `int`

Polymorphism

- New rule: when type-checking the use of a let-bound variable, create fresh versions of unconstrained type variables



- This rule is called ***let-based polymorphism***