A Note on Optimal Algorithms for Fixed Points

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22nd February 2010

Abstract

We present a constructive lemma that we believe will make possible the design of nearly optimal \(O(d \log \frac{1}{\epsilon})\) cost algorithms for computing \(\epsilon\)-residual approximations to the fixed points of \(d\)-dimensional nonexpansive mappings with respect to the infinity norm. This lemma is a generalization of a two-dimensional result that we proved in [1].

1 Introduction

In [1, 2] we presented two-dimensional optimal complexity algorithms for computing residual \(\epsilon\)-approximations to the fixed points of non-expansive mappings with respect to the infinity norm. These algorithms are based on bisection-envelope constructions and are derived from Theorem 3.1 of [1]. This theorem makes possible construction of a sequence of rectangles that contain fixed points and converge to the residual \(\epsilon\)-approximation of some fixed point. At every iteration of the process the previous rectangle is cut by a factor of at least two, to obtain a new rectangle containing a fixed point.

In this paper we generalize the constructive theorem to an arbitrary number of dimensions \(d \geq 3\), however, we are unable to utilize this new result in the construction of optimal algorithms.

The main obstacle in such construction is the ability to bound a new set containing fixed points by an “easy-to-construct” convex set of smaller volume and similar topological features to the previous set in this process. We stress that the two-dimensional sets in the optimal algorithm are rotated rectangles. What would be the proper sets in an arbitrary number of dimensions that would bound the non-convex sets resulting from the application of our general \(d\)-dimensional lemma?

2 Problem formulation

Given dimension \(d \geq 2\), we define \(D = [0, 1]^d\) and the class \(F\) of functions, \(f : D \rightarrow D\), that are Lipschitz continuous with constant 1 with respect to the
infinity norm, i.e.,
\[ \|f(x) - f(y)\| \leq \|x - y\| , \forall x, y \in D \]

where \( \|\cdot\| = \|\cdot\|_{\infty} \) henceforth. We seek an algorithm which, for every \( f \in F \), computes a solution \( \hat{x} = \hat{x}(f) \in D \) that satisfies the residual criterion
\[ \|f(\hat{x}) - \hat{x}\| \leq \epsilon \]  

(1)

where \( 0 < \epsilon < 0.5 \). (If \( \epsilon \geq 0.5 \) then \( x = (0.5, 0.5) \) satisfies [1]). The algorithm requires \( n(f) \) function evaluations, where \( n(f) \equiv O(d \log \frac{1}{\epsilon}) \). In the case of \( d = 2 \) the algorithm is based on Theorem 3.1 of [1], utilizes bisection of rectangles and envelope constructions, and has cost \( 2 \log_2 \frac{1}{\epsilon} \). Here we present a generalization of this theorem to the case of \( d \geq 3 \). We believe that the general result will provide the basis for construction of a future algorithm having the desired efficiency. So far we have been unable to construct such an algorithm. We stress that computing \( x_e, \|x_e - \alpha\| \leq \epsilon \), an \( \epsilon \)-absolute approximation to the fixed point \( \alpha \), in the class of expanding functions is of infinite complexity in the worst case [3].

3 Definitions

For a given \( f \in F \) and \( i = 1, \ldots, d \) we define the fixed point sets \( F_i \) such that for each \( i \),
\[ F_i(f) = \{ x \in D : f_i(x) = x_i \} . \]

We define \( F(f) = \bigcap_{i=1}^{d} F_i(f) \), the nonempty set of all fixed points of \( f \). For all \( x \in \mathbb{R}^d , i = 1, \ldots, d \), and \( s \in \{-1, 1\} \) we define the “open-ended” pyramid sets
\[ A^s_i(x) = \{ y \in \mathbb{R}^d : \|y - x\| = s(y_i - x_i) \} . \]

For all \( x \in \mathbb{R}^d , i = 1, \ldots, d \), \( s \in \{-1, 1\} \), and \( c > 0 \), we also define the “flat-top” pyramid set
\[ Q^s_i(x, c) = \cup \{ A^s_i(y) : y \in \mathbb{R}^d , \|y - x\| < c \} . \]

4 Constructive Lemma

In this section we prove our constructive lemma. It is a generalization of Theorem 3.1 of [1] to an arbitrary number of dimensions \( d \geq 3 \).

Lemma 4.1

For any \( f \in F_i , i = 1, \ldots, d \), we let \( x \in D \) be such that \( f_i(x) \neq x_i \). Then the following holds:

(i) If \( f_i(x) > x_i \) then \( Q^{-1}_i(x, (f_i(x) - x_i)/2) \cap D \cap F_i(f) = \emptyset \).
(ii) If $f_i(x) < x_i$ then $Q_i^1(x, (x_i - f_i(x))/2) \cap D \cap F_i(f) = \emptyset$.

**Proof.** To show (i) we take any $y$ such that $\|y - x\| < (f_i(x) - x_i)/2$, and $z \in A_i^{-1}(y) \cap D$. Then

$$|f_i(z) - f_i(y)| \leq \|f(z) - f(y)\| \leq \|z - y\| = y_i - z_i$$

and

$$f_i(y) - y_i = f_i(x) - (f_i(x) - f_i(y)) - x_i - (y_i - x_i) \geq f_i(x) - x_i - 2 \|y - x\|$$

$$> f_i(x) - x_i - (f_i(x) - x_i) = 0,$$

which implies

$$f_i(z) = f_i(y) + (f_i(z) - f_i(y)) > y_i - (y_i - z_i) = z_i.$$  

To show (ii) we take any $y$ such that $\|y - x\| < (x_i - f(x_i))/2$, and $z \in A_i^1(y) \cap D$. Then

$$|f_i(z) - f_i(y)| \leq \|f(z) - f(y)\| \leq \|z - y\| = z_i - y_i$$

and

$$f_i(y) - y_i = f_i(x) + (f_i(y) - f_i(x)) - x_i + (x_i - y_i) \leq f_i(x) - x_i + 2 \|y - x\|$$

$$< f_i(x) - x_i + (x_i - f_i(x)) = 0,$$

which implies

$$f_i(z) = f_i(y) + (f_i(z) - f_i(y)) < y_i + (z_i - y_i) = z_i. \quad \blacksquare$$

**Comments**

The above Lemma 4.1 states that after evaluating $f$ at $x$ we can remove from the original domain $D$ the “flat-top” pyramid sets $Q_i(x, c_i)$ for all $i$ such that $c_i = |f(x_i) - x_i|/2$ are not zero, since they do not contain fixed points of $f_i$, implying that they do not contain any fixed point of $f$ as well. If this happens for all $i = 1, \ldots, d$ then we can reduce the volume of the set containing fixed points by a factor of at least two.

**Open problems**

The main obstacle in constructing a recursive algorithm (for $d \geq 3$) based on Lemma 4.1 is our apparent inability to construct a sequence of sets $S_j$ that each contain a fixed point, are topologically “similar”, decrease in volume, and are easy to represent, and then evaluating $f$ at the “centers” of $S_j$. Also, it needs to be decided which sets can be removed from $S_j$ in the case where $f_i(x) - x_i = 0$, i.e., when the current evaluation point $x$ is a fixed point of some components of $f$.

We believe that by solving those problems we can obtain an optimal $O(d \log \frac{1}{\varepsilon})$ cost algorithm for finding $\varepsilon$-residual solutions to the fixed points of functions in our class. We hope to address these issues in a future paper.
References


