Lambertian Directional Sampling

Lambertian materials reflect light proportionally to $\cos\theta$, where θ is the angle between the reflected direction and the surface normal.

In spherical coordinates, this is expressed by the integral:

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}} k_\phi k_\theta \cos\theta \sin\theta d\theta d\phi = 1,$$

where k_ϕ and k_θ are constants that depend only on ϕ and θ, respectively. As with the uniform case we saw in class, this integral is separable and can be divided into the product of two integrals: one which only involves ϕ and one which only involves θ.

Solving for ϕ

The first integral is the same as we have seen before:

$$\int_0^{2\pi} k_\phi d\phi = 1.$$

Solving this for k_ϕ gives $k_\phi = \frac{1}{2\pi}$, which is the PDF for the (uniformly distributed) variable ϕ. Taking the integral of the PDF gives us the CDF for ϕ:

$$F(\phi) = \int_0^{\phi} \frac{1}{2\pi} dt = \frac{\phi}{2\pi}.$$

To generate a ϕ with this distribution, we solve the equation:

$$x = \frac{\phi}{2\pi},$$

which gives:

$$\phi = 2\pi x,$$

where x is a canonical uniform random variable, as obtained from `drand48()`, for example.

Solving for θ

The second integral is:

$$\int_0^{\frac{\pi}{2}} k_\theta \cos\theta \sin\theta d\theta = 1.$$

Solving this equation for k_θ gives $k_\theta = 2$, so the PDF for θ is:

$$q(\theta) = 2\cos\theta\sin\theta,$$

for θ on the interval $[0, \frac{\pi}{2}]$. As before, we take the integral to find the CDF:

$$Q(\theta) = \int_0^{\theta} 2\cos t \sin t dt = \sin^2\theta.$$

As with ϕ, we set this equal to the CDF for the uniform random variable x and solve for θ to get:

$$\theta = \arcsin\sqrt{x}$$