# Some History and Research of Frank Stenger

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## 1 The Earlier Years.

Frank Stenger was born in Magyarpolány, Hungary, on July 6, 1938. His parents were *Stenger György* and *Görgy Katalin*. An historical check shows that his mother*Görgy Katalin*'s ancestors were already present in Hungary in the early 1600's, whereas the Stenger side of the family came to Hungary in 1750, following the forced retreat of the Turks out of Hungary, by the Hapsburgs.

Both of Frank's grandparents, *György Miska* and *Stenger Simon* fought in WWI; *György Miska* was killed in that war, whereas *Stenger Simon* lost a leg and the use of an arm. Frank's father, *Stenger György*, spent nearly ten years conscripted by the Hungarian army preceding and during WWII. Indeed, he was one of the very few that survived after being captured on the *Russian Front*.

Magyarpolány was primarily a German speaking town, with a German dialect close to that of the street language of Vienna, although Hungarian was spoken equally often in the house in which Frank Stenger (who was called *Stenger Ferenc* in Hungary) lived since birth.

Frank Stenger's family was well off before the end of WWII, but they lost their home and all of their properties shortly after the end of that war. Frank, his sister, Kati (4 years younger than Frank) and his father and mother were forced to live with relatives, in cramped quarters, and in order to survive, they stole vegetables from the garden that they had previously planted on their former property.

In 1947 most of the people of Magyarpolány were ethnically cleansed out of

Hungary. They were locked into box cars and shipped off to East Germany. After spending 5 weeks in a refugee camp, the Stenger family was forced to live in the downstairs area of an older couple's house, in Zwönitz, Erzgb. Sachsen, East Germany. Whereas they were used to having plenty to eat in Hungary, food was scarce in East Germany at that time. For this, and for other reasons, after spending less than one year in East Germany, they sneaked over the border to West Germany. They lived in Braunschweig for about a year, after which Frank's uncle, who had moved to Alberta over 10 years earlier, arranged for them to come to Canada. They spent the first year in Warburg, Alberta, picking roots on Frank's uncle's farm, to pay back the money it cost his uncle to bring them to Canada.

Frank's father then bought a farm. He was, however, unlucky with crops: at the end of the first year of planting, all of the crops were lost due to hail; there was too much rain during the second year, which killed the crops; and the snow came too early in the third year, making it impossible to remove the crops in time, and as a result, they were eaten by mice over the winter. It was then that Frank decided that if he wanted to get an education, he would have to do it on his own. Consequently, he supported himself by winning scholarships, and fellowships and with excellent job opportunities until the end of his Ph.D. studies. His teachers, too, especially his high school mentor Erwin Stobbe, and his University of Alberta mentor, J.J. McNamee played important roles in these decisions.

Frank's sister, Kati (called Katherine in Canada) also did well scholastically. The two younger brothers, George and Edward, who were born in Canada did well monetarily, although they were less interested in scholastic achievements.

Frank owes a great deal for his education and academic career to:

His mother – for her many life-long sacrifices for him; His excellent teachers,

especially to his "Big Brother", Ervan Stobbe who started Frank's love of mathematics; and His mentor, John McNamee.

Frank enrolled in the electrical engineering program at the University of Alberta. While at residence, he met an Irish mathematician, John McNamee, who decided that Frank had some mathematical ability, and who gave Frank a copy of R. Courant's two-volume calculus texts. It did not take Frank long to solve all of the problems of these texts, which, in effect, sealed his fate in mathematics. After completion of his bachelor's degree in *Engineering Physics*, Frank enrolled in two simultaneous Master's degree programs, in Math. (Numerical Analysis, with specialty of n-dimensional quadratures), and in Engineering (Control Theory, with specialty of nonlinear controls). Although he completed the work of each, he only did a final oral and formal completion in his numerical analysis masters. He then did a Ph.D. degree in Math (Asymptotics), after spending a year at the National Bureau of Standards in Washington, D.C. His official Ph.D. adviser at the Univ. of Alberta Dept. of Math. was Ian Whitney, an expert in complex variables, who had previously worked with A. Erdélyi on the Bateman manuscript projects, and from whom Frank Stenger learned a great deal about analytic functions, including elliptic functions.

# 2 Some Research Results of Frank Stenger.

Only some of the research of Frank Stenger is touched upon in this report. There is, however, an (roughly) 80% complete set of references at the end of this report. Sinc related research of Frank Stenger is covered in the first subsection below. This is then followed by inverse problems research, miscellaneous research results, program packages, and textbooks.

## 2.1 Sinc Related Research of Frank Stenger.

1. In 1964, John McNamee and Ian Whitney wrote a joint paper, "Whittaker's Cardinal Function in Retrospect", which they submitted to SIAM. Unfortunately they had an incompetent referee, who made many unfair criticisms of their paper. They were, however proud men, and refused to rewrite their paper. McNamee gave a copy of this paper to Frank Stenger, and after reading it for the third time, Frank got quite excited about it. He approached McNamee and Whitney — offering to to rewrite the paper, by incorporating the two reasonable suggestions the referee had made, and also, to make some improvements, e.g., the inclusion of the Paley–Wiener theorem, more examples, as well as some applications — and if he did this, would they let him become a joint author. They agreed, and with this publication [16] the Cardinal function

$$
\begin{aligned}
C(f,h)(x) &= \sum_{k=-\infty}^{\infty} f(k\,h)\,S(k,h)(x) \\
S(k\,h)(x) &= \operatorname{sinc}\left(\frac{x}{h} - k\right) \\
\operatorname{sinc}(x) &= \frac{\sin(\pi\,x)}{\pi\,x}\,,
\end{aligned}
\tag{1}
$$

became a big part of Frank Stenger's approach to computation. Here, $\operatorname{sinc}(x)$ is the *sinc function* coined so by engineers, while we shall refer to the functions $S(k,h)$ as *Sinc functions*. The beautiful coinage of this function in the original paper (most likely due to McNamee) was "... a function of royal blood, whose distinguished properties separate it from its bourgeois brethren".

2. After his Ph.D. work, he spent 1965–66 in the Computer Science Department at the University of Alberta, In the fall of 1966, he joined the Mathematics Department at the University of Michigan as an Assistant professor, where he concentrated primarily on improving his mathematical skills. At that time, there were over 20 seminars held in the department each week, and he attended 11 of these. One of the themes of the department involved various aspects of the Wiener-Hopf process: the applied mathematicians were solving Wiener–Hopf problems; the functional analysts were factoring Toeplitz operators; the approximation group was studying approximation by rations of analytic functions; and the probabilist were studying discrete Wiener–Hopf equations. In particular, during that time, *Ron Douglas* and *Walter Rudin* wrote a joint paper, proving that: *Given function* $f \in \mathbf{L}^\infty(T)$ *, with* $T$ *the unit circle, given any* $\varepsilon > 0$*, there exists a positive integer* $n$ *, inner functions* (functions that are analytic and unimodular in the unit disc $U$) $\varphi_1\,,\ \psi_1\,,\ \ldots \varphi_n\,,\ \psi_n$ *and constants* $c_1\,,\ \ldots\,,\ c_n$ *such that*

$$
\operatorname*{ess\,sup}_{r \to 1^-} \left| f\left(e^{i\,\theta}\right) - \sum_{j=1}^{n} c_j\,\frac{\varphi_j\left(r\,e^{i\,\theta}\right)}{\psi_j\left(r\,e^{i\,\theta}\right)} \right| < \varepsilon \tag{2}
$$

*a.e.*

Stenger [17] gave a constructive proof of this result, showing moreover, that the approximation can be accomplished with $n = 2$. In his proof, Stenger constructed a novel elliptic function, which was an approximate characteristic function on a measurable set. Shortly afterward, while visiting the University of Montreal in 1970, he reconstructed a variant of this beautiful function in great detail [38], based on a geometric-analytic function proof, via use of elliptic functions. Indeed, Stenger considers [38] to be his first *Sinc methods* paper, which led him to the development of the area of Sinc computation. In the same paper, [38] he disproved a conjecture made earlier by several others, which we now describe. In 1965 [130], H.S. Wilf posed the problem, to determine the best $\sigma(n)$, with

$$\sigma(n) =$$

$$\inf_{w_j \in \mathbb{C},\ z_j \in U} \left\{ \sup_{f \in \mathbf{H}^2(U), \|f\|=1} \left| \int_{-1}^{1} f(x)\, dx - \sum_{j=1}^{n} w_j\, f(x_j) \right| \right\}, \tag{3}$$

where $\mathbf{H}^2(U)$ is the Hardy space of all functions $f$ that are analytic on the unit disc $U$, normed by

$$\|f\| = \left( \lim_{r \to 1^-} \int_0^{2\pi} \left| f\left(r\, e^{i\,\theta}\right) \right|^2 d\theta \right)^{1/2}.$$

In his paper [130], Wilf obtained

$$\sigma(n) = \mathcal{O}\left( \left( \frac{\log(n)}{n} \right)^{1/2} \right).$$

Shortly thereafter there followed three independent research articles, each of which obtained the estimate $\sigma(n) = \mathcal{O}(n^{-1/2})$: by S. Haber (in Quart. Appl. Math., 29 (1971) 41-420), by Johnson & Riesz (University of Toronto Research Report) and S. Ecker (Ph.d. thesis, University of Hamburg) and with each independently making the conjecture that this is the best bound possible. Stenger got $\sigma(n) = \mathcal{O}\left(\exp\left(-\pi\,(n/2)^{1/2}\right)\right)$, via use of an explicit "$q$–series" type quadrature formula constructed in [38].

In [38][1] Stenger obtained several explicit $q$-series–type quadrature formulas via transformations of simple conformal maps in the characteristic function he constructed: (i) an arc of the unit circle, (ii) the interval $(-1,1)$, (iii) the interval $(0,\infty)$, and (iv) the real line $\mathbb{R} = (-\infty\,,\infty)$. The latter was just

[1] The publication of this paper came much later than its discovery. After the referee of the journal it was submitted to kept it for over a year, he recommended turning it down.

the trapezoidal rule, but with nearly the exact same bound on the error that was previously obtained in [16]. This connection with the above Cardinal series led his mentor, J. McNamee, to suggest to Stenger that he should use this Cardinal series rather than elliptic functions to derive such formulas, since the Cardinal series would be understood by a wider audience.

3. As a visitor to the Univ. of British Columbia during the academic year of 1975–76, Stenger applied various conformal maps to the series (1), to get explicit, accurate Sinc–type methods for approximation over arbitrary intervals and contours [34].

4. Whenever the series (3) above converges, the resulting function is an entire function of order 1 and type $\pi/h$ . If $f$ is also an entire function of order 1 and type $\pi/h$ that is uniformly bounded on $\mathbb{R}$ , then the function $C(f,h)$ defined in (1) above satisfies the identity $C(f,h) = f$ . In this space the function $C(f,h)$ is replete with many identities obtained via operations on $C(f,h)$, such as differentiation, orthogonality, delta function-like behavior of Sinc functions, Fourier transforms, Hilbert transforms, etc. These identities become highly accurate approximations if $f$ is not analytic in the entire complex plane, but rather, analytic and uniformly bounded only in the strip

$$D_d = \{z \in \mathbb{C} : |\Im(z)| < d\} \, ,$$

a region which arose naturally in the derivation of the quadrature rules of [38]. A conformal map $\varphi$ of another region

$$\mathcal{D} = \{z \in \mathbb{C} : |\arg(\varphi(z))| < d\}$$

onto $D_d$ automatically yields methods of interpolation (as well as other formulas of approximation) over a contour $\Gamma = \varphi^{-1}(\mathbb{R})$, of the form

$$\begin{aligned} F &\approx C(F,h)\circ\varphi = \sum_{k=-\infty}^{\infty} F(z_k)\, S(k,h)\circ\varphi \, , \\ \int_\Gamma F(x)\, dx &\approx \int_\Gamma C(F,h)\circ\varphi(x)\, dx \approx h \sum_{k=-\infty}^{\infty} \frac{F(z_k)}{\varphi'(z_k)} \, . \end{aligned} \tag{4}$$

with $z_k = \varphi^{-1}(k\,h)$ denoting the *Sinc points*. Moreover, those same identities for the function $C(f,h)$ now still hold, and one gets exactly the same bounds on the errors of approximation. In this way, we get an explicit family of formulas for interpolation, quadrature, differentiation, and Hilbert transforms, ..., etc., for arbitrary bounded, semi–infinite, infinite intervals and even for analytic arcs.

For example: If $\varphi(z) = \log(z)$, then $\mathcal{D}$ is the sector, $|\arg(z)| < d$, the interval $\Gamma = (0, \infty)$, the Sinc points are $z_k = e^{kh}$, the "weights", $h/\varphi'(z_k) = h\, e^{kh}$;

If $\varphi(z) = \log(z/(1-z))$, then $\mathcal{D}$ is the "eye-shaped" region (see [4], p. 68, for a picture), $\{|\arg(z/(1-z))| < d\}$, the interval $\Gamma = (0, 1)$, the Sinc points are $z_k = e^{kh}/(1 + e^{kh})$, the "weights", $h/\varphi'(z_k) = h\, e^{kh}/(1 + e^{kh})^2$.

5. The concept of *Sinc spaces* was formulated somewhat later (in 1984). An understanding of these spaces enable one to tell *a priori* when one can achieve uniform accuracy over $\Gamma$ via use of Sinc methods, by means of a relatively small number of points. (Actually, because the Sinc methods approximate functions at Sinc points – all of which are in the interior of $\Gamma$ – they achieve approximations that are accurate to within a *relative error* even when they approximate an operation the result of which is unbounded at an end point of $\Gamma$. This occurs e.g., for differentiation, for Laplace transform inversion, for the approximation of Hilbert transforms, for the approximation of Abel–type integrals, etc.) It is convenient to introduce the Sinc spaces at this time.

Along with the conformal map $\varphi$ of $\mathcal{D}$ to $D_d$, we set $\rho = \exp(\varphi)$, we denote the end points of $\Gamma = \varphi^{-1}(\mathbb{R})$ by $a = \varphi^{-1}(-\infty)$ and $b = \varphi^{-1}(\infty)$, we assume that $F$ is analytic and bounded in $\mathcal{D}$, and that limiting values $F(a)$ and $F(b)$ exist, so that the expression

$$\mathcal{L}F = \frac{F(a) + \rho\, F(b)}{1 + \rho} \tag{5}$$

is well defined. (Note that as traverses $\Gamma$ from $a$ to $b$, $\rho(z)$ is real valued, and increases strictly from 0 to $\infty$.) We then say that

A. $F \in \mathbf{L}_{\alpha,d}(\varphi)$ *if there exist positive constants* $C$ *and* $\alpha$ *such that* $|F(z)| < C \exp(-\alpha\, |\varphi(z)|)$ *for all* $z \in \mathcal{D}$.

B. $F \in \mathbf{M}_{\alpha,d}(\varphi)$ *if* $F - \mathcal{L}F \in \mathbf{L}_{\alpha,d}(\varphi)$.

Let us also introduce the *Hilbert* and *Cauchy* transforms:

$$\begin{aligned}(\mathcal{S}\,F)(\tau) &= \frac{P.V.}{\pi\,i}\int_\Gamma \frac{F(t)}{t-\tau}\,dt \quad \tau\in\Gamma,\\ (\mathcal{C}\,F)(z) &= \frac{1}{2\,\pi\,i}\int_\Gamma \frac{F(t)}{t-z}\,dt\,, \quad z\notin\Gamma.\end{aligned} \tag{6}$$

The following properties hold for Sinc spaces:

**Theorem:** [87, §1.4] Let $\alpha\in(0,1]$, $d\in(0,\pi]$, and take $d'\in(0,d)$.

**i.** If $F\in\mathbf{M}_{\alpha,d}(\varphi)$, then $F^{(n)}/(\varphi')^n\in\mathbf{L}_{\alpha,d'}(\varphi)$, $n=1,2,3,\cdots$;

**ii.** If $F'/\varphi'\in\mathbf{L}_{\alpha,d}(\varphi)$, then $F\in\mathbf{M}_{\alpha,d}(\varphi)$;

**iii.** If $F\in\mathbf{L}_{\alpha,d}(\varphi)$, then $\int_\Gamma|\varphi'(x)\,F(x)\,dx|<\infty$;

**iv.** If $F\in\mathbf{L}_{\alpha,d}(\varphi)$, then both $\mathcal{S}\,F$ and $\mathcal{C}\,F$ belong to $\mathbf{M}_{\alpha,d'}(\varphi)$.

These spaces are connected, in that, e.g., if $\varphi_1:\mathcal{D}_1\to D_d$ and $\varphi_2:\mathcal{D}_2\to D_d$ are two conformal maps, and if $F\in\mathbf{M}_{\alpha,d}(\varphi_1)$ then $F\circ\varphi_1^{-1}\circ\varphi_2\in\mathbf{M}_{\alpha,d}(\varphi_2)$.

6. The truncation of infinite Sinc series to a finite ones became well established by this time, as well as a slight alteration of the bases from $S(k,h)\circ\varphi$ to $\omega_k$ (see §1.4 of [10]) which enabled uniformly accurate Sinc approximation over $\Gamma$ of functions which are bounded but non-zero at the end-points of $\Gamma$. An added bonus arose with this truncation: Suppose that $F$ defined on $\mathbb{R}$ is analytic and bounded in $D_d$, and that, with $L\,F$ defined in (5) above, $F(z)-(\mathcal{L}\,F)(z)=\mathcal{O}(\exp(-\alpha\,|z|))$ on $\mathcal{D}$, which translates to $F\in\mathbf{M}_{\alpha,d}(\mathrm{id})$, with id the identity map. If $\varphi$ is a conformal map of $\mathcal{D}$ to $D_d$, with $\varphi:\Gamma\to\mathbb{R}$, then $G=F\circ\varphi$, which belongs to the class $\mathbf{M}_{\alpha,d}(\varphi)$ is not only analytic and bounded on $\mathcal{D}$, but it furthermore belongs to $\mathbf{Lip}_\alpha(\Gamma)$. The class $\mathbf{M}_{\alpha,d}(\varphi)$ thus houses solutions of differential equations, and we get exponential convergence when approximating such functions, even though we don't know the exact nature of the singularities at end–points of intervals (or on the boundary of a region in more than one dimension, when solving PDE).

For example, if $d$ and $\alpha$ are some positive constants, then the choice $h=c'/N^{1/2}$, with $c'$ an arbitrary positive constant, independent of $N$, $d$ or $\alpha$ yields an error of the form

$$\sup_{x\in\mathbb{R}}\left|F(x)-\sum_{k=-N}^{N}F(z_k)\,\omega_k(x)\right| = \mathcal{O}(\exp(-c\,N^{1/2}))\,,\quad N\to\infty\,. \tag{7}$$

with $c$ a positive constant. The best (i.e., largest) $c=\sqrt{\pi\,d\,\alpha}$, which obtains with $h=\sqrt{\pi\,d/(\alpha\,N)}$.

7. Also during his visit to the Univ. of British Columbia in '75–'76, Stenger discovered the important Sinc indefinite integration matrices, for approximating the operations

$$(\mathcal{J}^+\,g)(x)=\int_a^x g(t)\,dt\,,\quad\text{and}\quad(\mathcal{J}^-\,g)(x)=\int_x^b g(t)\,dt\,. \tag{8}$$

By this time, it became understood, that Sinc methods are easily dealt with via matrix techniques, inasmuch as the basis functions only generate the matrices for approximating operations of calculus, whereas we are interested only in vectors of values of functions at Sinc points.

Letting $u$ denote an arbitrary function defined on $\Gamma$, and letting $\omega_j$ denote the Sinc basis as defined in §1.4 of [87] it is now convenient to define a diagonal matrix $D(u)$, an operator $V$ that changes $u$ to a column vector $V(u)$, and a row vector $\mathbf{w}$ of Sinc basis functions on $\Gamma$ by

$$\begin{aligned} D(u) &= \operatorname{diag}[u(z_{-N}),\ldots,u(z_N)] \\ V(u) &= (u(z_{-N}),\ldots,u(z_N))^T \\ \mathbf{w} &= (\omega_{-N},\ldots,\omega_N)\,. \end{aligned} \tag{9}$$

We also need certain numbers, $\sigma_k$, generated by integrals of Sinc functions. These are given by

$$e_k=\int_0^k \operatorname{sinc}(x)\,dx\,,\qquad \sigma_k=1/2+e_k\,,\qquad k=0\,,\ \pm1\,,\ \pm2\,,\ \ldots\,.$$

Let $m=2\,N+1$, and let $I_m^{(-1)}$ denote[2] $m\times m$ matrix with $(i,j)^{th}$ element $\sigma_{i-j}$. Then setting

[2]To date it has been shown via numerical computation that all eigenvalues of $I_m^{(-1)}$ lie in the open right half plane, for $1\le m\le 1024$. Stenger offers $300 to the first person who proves or disproves that all of the eigenvalues of $I_m^{(-1)}$ lie in the open right half plane, for every positive integer $m$.

$$A^{+} = h\, I_m^{(-1)}\, D(1/\varphi')\,, \quad A^{-} = h\, \left(I_m^{(-1)}\right)^T D(1/\varphi')\,, \tag{10}$$

we get the accurate approximations

$$\begin{aligned} (\mathcal{J}^{+} g)(x) &= \int_a^x g(t)\, dt &\approx& \ \mathbf{w}(x)\, A^{+}\, V\, g \\ (\mathcal{J}^{-} g)(x) &= \int_x^b g(t)\, dt &\approx& \ \mathbf{w}(x)\, A^{-}\, V\, g\,. \end{aligned} \tag{11}$$

Although this formula was discovered in 1976-77, it was first published without proof only in 1981 [58]. Several proofs have been given since.

8. One important application of Sinc indefinite integration is that it enables uniform approximation of indefinite integrals on arbitrary intervals and contours, even when the integrands are unbounded at (but integrable over $\Gamma$) at end–points of $\Gamma$. (As already mentioned above, we can even get good approximations of integrals that are unbounded, e.g., the incomplete Gamma function, $\Gamma(a,x) = \int_x^\infty t^{a-1}\, dt$, with $a \le 0$). Another important application was a novel package for solving ODE (ordinary differential equation) initial value problems [92] over arbitrary intervals; furthermore, while *stability* and *stiffness* can cause difficulties for other methods and packages, these are not difficulties for the Sinc ODE package. The reasons: The above approximation to $\mathcal{J}^{+}$ is applied after the usual conversion of the ODE to one, or a system of integral equations, enabling an immediate reduction to a system of algebraic equations, via Sinc collocation over the whole interval, a procedure that requires no computation. The resulting system of algebraic equations is then solved via Newton's method, thereby avoiding problems of stability. Stiffness is not a problem since the Sinc points "bunch up" at the end–points of the interval. and are thus able to accurately approximate solutions even though they may change rapidly in neighborhoods of the end-points.

9. It was also shown in 1984, that if the coefficients of function values at the Sinc points are in error, then the error of Sinc interpolation on $\Gamma$ is not appreciably larger in magnitude than the error in the coefficients. This fact and the fact that Sinc collocation is equivalent to Sinc–Galerkin enables us to drop the sinc basis vector $\mathbf{w}$ in our Sinc approximation formulas.

10. In [67] Elliott & Stenger obtained the following formula for approximating the above Cauchy integral $\mathcal{C}F$:

$$
\begin{aligned}
(\mathcal{C}\, F)(z) &\approx \sum_{k=-N}^{N} F(z_k)\, c_k(z)\,, \\
(\mathcal{S}\, F)(\tau) &\approx \sum_{k=-N}^{N} F(z_k)\, t_k(\tau)\,,
\end{aligned}
\tag{12}
$$

with

$$
\begin{aligned}
c_k(z) &= \frac{h}{2\,\pi\, i}\, \frac{\exp\{i\,\pi\, [\varphi(z) - kh]/h\} - 1}{\varphi'(z_k)(z - z_k)}\,, \\
t_k(\tau) &= \frac{h}{\pi}\, \frac{\cos\{\pi[\varphi(x) - kh]/h\} - 1}{\phi'(z_k)(x - z_k)}.
\end{aligned}
\tag{13}
$$

11. This is a stable method of analytic continuation, i.e., in determining the values of an analytic function in the interior of a domain once they are known on Sinc points of the boundary. Additionally, the formula (12) combined with an equally efficient variant of (12) for Sinc approximation of Hilbert transforms (see below, however, for a recently discovered variant that is even better!) were used in [106] for devising an (i.e., probably the most efficient) algorithm to construct conformal maps of regions whose boundary consists of a finite number analytic arcs.

12. A family of rational functions was constructed in the paper [73], which interpolated functions at Sinc points $z_{-N}\,,\ \ldots\,,\ z_N$ and for which the error of interpolation of functions belonging to the above Sinc spaces had the same bound as the error of Sinc interpolation at these same points. The most important consequence of this was in the area of rational extrapolation to the limit. If a function $f$ is analytic in a simply connected domain $\mathcal{D}$, and if points $z_1\,,\ z_2\,,\ \ldots\,,\ z_n$ and $\zeta$ are in $\mathcal{D}$ then the known values of $f$ at the points $z_j$ can be used to predict the value $f(\zeta)$ via use of polynomial extrapolation, and moreover, the process converges exponentially, with error of the order of $\mathcal{O}(\exp{(-c\,n)}$. The reason for this is that there exists a polynomial of degree $n$ for which the maximum difference between $f$ and the polynomial in a simply connected compact subset of $\mathcal{D}$ is of the order of $\exp{(-c\,n)}$. However, if $\zeta$ is on the boundary of $\mathcal{D}$ such that $f$ has an algebraic singularity at $\zeta$, then the convergence of polynomial extrapolation is so slow making polynomial extrapolation practically worthless.

On the other hand, if the above points $z_j$ belong to a domain $\mathcal{D}$ which is mapped by $\varphi$ onto the above defined strip $\mathcal{D}_d$, if $f \in \mathbf{M}_{\alpha,d}(\varphi)$, and if $\zeta$ either belongs to $\mathcal{D}$ or is an end-point of $\Gamma = \varphi^{-1}(\mathbb{R})$, then there is a rational function of the type derived in [73] which interpolates $f$ at $2N+1$ points of $\Gamma$ and for which the maximum difference between $f$ and the rational at the points $z_j$ and at $\zeta$ is of the order of $\exp\left(-c\,n^{1/2}\right)$, i.e., we can be sure that rational extrapolation works to predict the value of $f(\zeta)$, even though this value might be difficult or impossible to compute directly. That is whereas the success of rational extrapolation was previously based on a "gut feeling', we now conditions under which it is guaranteed to work. Some examples of this type are given in [87]; other practically important ones, including not yet tried possibilities are given below.

13. A recent discovery of Stenger is the formula for indefinite convolutions [89], which has led to many novel important formulas in applications, for approximating convolution integrals, for inverting Laplace transforms, for solving integral equations, such as Wiener–Hopf equations, which were hitherto considered to be difficult, for evaluating Hilbert transforms, for solving PDE (partial differential equations) and for solving multidimensional integral equations. The basic models are the integrals

$$\begin{aligned} p(x) &= \int_a^x f(x-t)\, g(t)\, dt\,, \\ q(x) &= \int_x^b f(t-x)\, g(t)\, dt\,, \end{aligned} \tag{14}$$

with $x \in (a,b)$ (i.e., this process has not yet been studied for a more general contour $\Gamma$.) The "Laplace transform"

$$\mathcal{F}(s) = \int_E f(t)\, e^{-t/s}\, dt \tag{15}$$

is required, with $E$ any subinterval of $\mathbb{R} = (-\infty,\infty)$ such that $E \supseteq (0, b-a)$, exists for all $s \in \Omega_+ \equiv \{s \in \mathbb{C} : \Re s > 0\}$. It is shown in [89] that

$$p = \mathcal{F}(\mathcal{J}^+)\, g \quad \text{and} \quad q = \mathcal{F}(\mathcal{J}^-)\, g\,, \tag{16}$$

with $\mathcal{J}^+$ and $\mathcal{J}^-$ defined as in (11) above.

Using the approximations of (11), i.e., $\mathcal{J}^{\pm}\, g \approx \mathcal{J}^{\pm}_m\, g$ with $\mathcal{J}^{\pm}_m = \mathbf{w}\, A^{\pm}\, V$, and with $V$ defined as in (9) above, one can surmise that if $\mathcal{J}^{\pm} \approx \mathcal{J}^{\pm}_m$, then $\mathcal{F}(\mathcal{J}^{\pm}) \approx \mathcal{F}(\mathcal{J}^{\pm}_m)$, and indeed, this was shown to be the case in [89]. One thus gets the approximations

$$V\,p \approx F(A^+)\,V\,g \quad \text{and} \quad V\,q \approx F(A^-)\,V\,g \tag{17}$$

with the error these approximations of the same order as the error in (6) above.
Here, the matrices $F(A^\pm)$ can be evaluated by diagonalization of $A^\pm$, (a procedure which has always been possible numerically, to date) for example, if $A^+ = X\,S\,X^{-1}$, with $S = \text{diag}\,(s_{-N}\,,\,\ldots\,,\,s_N)$, then $F(A^+) = X\,F(S)\,X^{-1}$.

14. Summarizing to this point, we see that Sinc methods offer a self contained family of approximations of the most significant operations of calculus, and with the error being of the order of that in (6) above:

*Interpolation:* $F \in \mathbf{M}_{\alpha,d}(\varphi) \Longrightarrow$

$$V\,F = V\,F \qquad (F(x) \approx \mathbf{w}(x)\,V\,F)\,;$$

*Differentiation:* $\varphi'\,F \in \mathbf{L}_{\alpha,d}(\varphi) \Longrightarrow$

$$V\,F' \approx (A^+)^{-1}\,V\,F \quad (\text{or} \quad V\,F' \approx -(A^-)^{-1}\,V\,F\,;$$

*Indefinite Integration:* $F/\varphi' \in \mathbf{L}_{\alpha,d}(\varphi) \Longrightarrow$

$$V\,\mathcal{J}^\pm\,F \approx A^\pm V\,F\,;$$

*Quadrature:* $F/\varphi' \in \mathbf{L}_{\alpha,d}(\varphi) \Longrightarrow$

$$\int_\Gamma F(x)\,dx \approx h\,(V\,f)^T\,V(1/\varphi')\,;$$

*Indefinite Convolution:* (See above for the definitions of $p$ and $q$.) More complicated precise conditions hold for the following, but usually it suffices if $p$ and $q$ belong to $\mathbf{M}_{\alpha,d}(\varphi)$.

$$V\,p \approx F(A^+)\,V\,g\,, \qquad V\,q \approx F(A^-)\,V\,g\,;$$

*Hilbert Transform:* See (6) & (12) above. Uniform error bounds for the following hold if $f \in \mathbf{L}_{\alpha,d}(\varphi)$:

$$V\,\mathcal{S}\,F \approx \left(\log(A^-) - \log(A^+)\right)\,F\,;$$

This result was derived recently, in [10] .

*Cauchy Transforms.* See (6) & (12) above. Uniform error bounds for the following hold if $f \in \mathbf{L}_{\alpha,d}(\varphi)$ .

$$(\mathcal{C}\, F)(z) \approx \sum_{k=-N}^{N} F(z_k)\, c_k(z)\,, \quad z \notin \Gamma;$$

*Laplace Transform Inversion:* If $g(s) = \int_0^\infty f(t)\, dt$ , then (over an interval $\Gamma$ , with $\varphi : \Gamma \to \mathbb{R}$), and if $\mathbf{1}$ is a column vector of order $m$ with a "1" in each entry, then

$$V\, f \approx \left(A^+\right)^{-1}\, g\left(\left(A^+\right)^{-1}\right)\, \mathbf{1}\,;$$

Uniformly accurate error bounds hold for this approximation if $f \in \mathbf{M}_{\alpha,d}(\varphi)$ .

*Inner Product Evaluations in Galerkin Methods.* If $f/\varphi' \in \mathbf{L}_{\alpha,d}(\varphi)$ , and if $\omega_j$ denotes the $j^{th}$ basis function, then

$$\int_\Gamma f(x)\, \omega_j(x)\, dx \approx \frac{h\, f(z_j)}{\varphi'(z_j)}\,.$$

15. The above one dimensional convolution procedure extends readily to multidimensional convolutions. And while the approximation of one dimensional convolutions requires the diagonalization of a matrix and thus seemingly requires more work than we are normally used to expend on a numerical method, this amount of work is relatively small for solution of partial differential equations.

16. The solution of PDE that are expressed via integrals of Green's functions is a straight forward application of Sinc convolution, but to achieve this, one requires the multidimensional "Laplace transform" of the Green's function. Stenger was lucky in this endeavor, in that he was able to obtain explicit expressions of all of the free space multidimensional Green's functions known to him for Poisson, biharmonic, wave, and heat problems. The derivations of these are given in [121 ]. Also in [121], Stenger gives explicit algorithms for the evaluation of the Green's function convolution integrals, first over rectangular, and then also over curvilinear regions. It has thus become possible to achieve a highly efficient and accurate approximation of multidimensional Green's function convolution integrals via the use of a very

small number of multiplications of one dimensional matrices i.e., via *separation of variables*. We are thus able to circumvent the use of large matrices required via use of classical finite difference or finite element methods, and thus, to get uniformly accurate solution via use of considerably less effort.

For example, to evaluate the solution of a Poisson problem for a function $U$, over a planar region $B$, with

$$U(x,y) = \int\int_B \mathcal{G}(x-\xi, y-\eta)\, e(\xi,\eta)\, d\xi\, d\eta\,.$$
$$\mathcal{G}(x,y) = \frac{1}{2\pi}\,\log\frac{1}{\sqrt{x^2+y^2}}\,, \tag{18}$$

and with $e$ a forcing function that might be unbounded on the boundary of $B$ but is integrable over $B$, we need the two dimensional "Laplace transform" $\hat{G}$ of the Green's function $\mathcal{G}$, i.e.,

$$\hat{G}(u,v) = \int_0^\infty\int_0^\infty \exp\left(-\frac{x}{u}-\frac{y}{v}\right)\,\mathcal{G}(x,y)\,dx\,dy$$
$$= \left(\frac{1}{u^2}+\frac{1}{v^2}\right)^{-1}\cdot \tag{19}$$
$$\cdot\left(-\frac{1}{4}+\frac{1}{2\pi}\left(\frac{v}{u}\,(\gamma-\ln(v))+\frac{u}{v}\,(\gamma-\ln(u))\right)\right)\,.$$

Thus, if $B$ is a rectangular region such as $B = (0,1)\times(0,1)$, if $A^\pm$ are the Sinc indefinite integration matrices of order $m = 2N+1$ over $(0,1)$, with $A^+ = X\,S\,X^{-1}$, $A^- = Y\,S\,Y^{-1}$, with $S$ a diagonal matrix, $S = \mathrm{diag}(s_{-N}\,,\,\ldots\,,\,s_N)$, with $Xi = X^{-1}$ and $Yi = Y^{-1}$, with $G$ the matrix with $(i,j)^{th}$ entry $\hat{G}(s_i\,,s_j)$, and with $E$ the matrix with $(i,j)^{th}$ entry $e(z_i,z_j)$, and where the $z_k$ denote Sinc points, we can approximate $U(z_i,z_j)\approx U_{ij}$ via use of the following *Matlab* program:

$$U = X*(G.*(Xi*E*Xi.'))*X.';$$
$$U = U+Y*(G.*(Yi*E*Xi.'))*X.';$$
$$U = U+X*(G.*(Xi*E*Yi.'))*Y.';$$
$$U = U+Y*(G.*(Yi*E*Yi.'))*Y.';$$

This approximate solution has a uniform error of the order of that on the right hand side of (6), provided that each of the functions, $e(\cdot,y)$ for each fixed $y \in (0,1)$ and $e(x,\cdot)$ for each fixed $x \in (0,1)$ are analytic, and provided that $U$ is uniformly bounded on $B$.

We may note that this procedure just involves the product of a few one dimensional matrices (separation of variables!). Similarly, higher dimensional problems, including, e.g., problems over a rectangular region $B$ in $\mathbb{R}^3$, or over $B \times (0,T)$ are not much more difficult to solve.

17. For example, the electric field integral equation

$$\mathbf{e}(\mathbf{r},t) - \int_V \int_0^t \left( \int_0^{t'} \gamma(\mathbf{r}', t' - \xi)\mathbf{e}(\mathbf{r}', \xi) d\xi \right) g(\mid \mathbf{r} - \mathbf{r}' \mid, t - t') dt' d^3\mathbf{r}' = \mathbf{e}^{in}(\mathbf{r}, t)$$

was solved in [125] after collocation via Sinc convolution and then solution of the resulting system of equations via successive approximation. Naghsh-Nilchi obtained the following computation times:

**IBM RISC/560 Workstation Run–Times**

Computation time required by Yee's Finite Difference (F.D.) and *Sinc-convolution* methods vs. desired precision. Computer run–time is shown as Days: Hours: Minutes: Seconds

The unstarred entries are actual computation times.

The starred $(\{\cdot\}^*)$ entries are computed computation times, based on known rates of convergence of the finite difference method of Yee [128]. The unstarred entries are actual computation times.

| **Acc.** | **F. D. Run-Time** | **Sinc–Conv. Run-Time** |
|---|---|---|
| $\mathbf{10^{-1}}$ | $\approx$ 1 second | $\approx$ 1 second |
| $\mathbf{10^{-2}}$ | 000:00:00:27 | 000:00:00:06 |
| $\mathbf{10^{-3}}$ | 003:00:41:40* | 000:00:02:26 |
| $\mathbf{10^{-4}}$ | > 82 years* | 000:00:43:12 |
| $\mathbf{10^{-5}}$ | > 800,000 years* | 000:06:42:20 |
| $\mathbf{10^{-6}}$ | > 8.2 billion years* | 001:17:31:11 |

18. Many other PDE have been solved since via the Sinc convolution approach, each illustrating the ease of use and accuracy of Sinc methods. The tutorial [121] contains examples of the solution of PDE over curvilinear regions, solution of nonlinear PDE, one of which (the nonlinear integro–differential equation in [10, §4.3.1]) no-one else was able to solve via any other method, and the ease of solution of wave and time problems via use of Neumann type iteration, which works, in essence for Sinc methods whenever it works in theory.

19. In [17] Morse & Feshbach discuss the possibility of use of separation of variables to solve three dimensional Laplace and Helmholtz equations. They conclude via use of the *Stäckel* determinant, that there are essentially only 13 coordinate systems for which this is possible. The key to success of this procedure is, in essence is to be able to transform the problem over the original region into a similar one over a rectangular region. They would then be able to use one-dimensional methods to solve multidimensional problems. Stenger shows in [10] that such separation of variables is possible for Poisson, wave and heat problems in all dimensions. One reason for this is that the Sinc methods of this package enable solutions of PDE without approximation of the highest derivatives. The procedure is based on the above convolution method. One does, however, require one additional property for success, which is that the coefficients of PDE as well as the patches of the boundary of the region are analytic in each variable, with all other variables held fixed,

and real. PDE from applications that are modeled via use of calculus do, in fact have this feature, under the assumption that such PDE are modeled by scientists and engineers, via use of calculus. In such circumstances, Sinc methods also yield exponential convergence, and combined with the separation of variables referred to above, we are able to obtain significant increases in the rates of convergence over classical methods.

Stenger has also demonstrated that this separation of variables procedure extends to PDE defined over curvilinear regions, $B$, under the assumption that such regions $B$ can be represented as a union of rotations of a finite number of regions of the form

$$\mathcal{B} = \{(x,y) : a_1 < x < b_1,\ a_2(x) < y < b_2(x)\} \tag{20}$$

in two dimensions, and of the form

$$\mathcal{B} = \{(x,y,z) : a_1 < x < b_1,\ a_2(x) < y < b_2(x),\ a_3(x,y) < z < b_3(x,y)\}\ , \tag{21}$$

in three dimensions (and similarly, in more than 3 dimensions). These regions can be transformed into rectangular ones via the respective transformations

$$\begin{aligned} x &= a_1 + (b_1 - a_1)\xi \\ y &= a_2(x) + (b_2(x) - a_2(x))\,\eta\,, \end{aligned} \tag{22}$$

and

$$\begin{aligned} x &= a_1 + (b_1 - a_1)\,\xi \\ y &= a_2(x) + (b_2(x) - a_2(x))\,\eta \\ z &= a_3(x,y) + (b_3(x,y) - a_3(x,y))\,\zeta\,. \end{aligned} \tag{23}$$

These transformations map these regions to rectangular ones while preserving the requisite analyticity property referred to above, provided that the $a_i$ and $b_i$ have similar analyticity properties. Similar assumptions must be made of the boundary, which is to consist of a finite number of analytic arcs in two dimensions, and of a finite number of "analytic patches" in three dimensions[3]. Surprisingly, (i.e., since such transformations "disturb" the convolutions, although luckily, they do not disturb them enough) we are

[3]Stenger makes these concepts precise in [121].

still able to use separation of variables to evaluate the convolution integrals after such transformations are made, via use of the original "Laplace transformed" Green's functions.

20. Success of the above procedure also depends on recently obtained novel extensions in [121] of the one dimensional convolution formulas for approximating the above one dimensional integrals $p$ and $q$ in (14) , to the approximation of integrals of the form

$$\begin{aligned} r(x) &= \int_a^x g(x-t,\, t)\, dt\,, \\ s(x) &= \int_a^x k(x\,, x-t\,, t)\, dt\,. \end{aligned} \tag{24}$$

21. Stenger also shows in [121] that if the non-homogeneous term of the PDE has such analyticity properties, then so does the result of a convolution of this term with a Green's function, and this enables him to show that the solution of the PDE also has the correct analyticity properties to enable the achievement of exponential convergence at a rate (7) in the approximate solution.

## 2.2 Some Inverse Problem Results of Frank Stenger.

The company, *TechniScan Inc.*, was founded by S.A. Johnson, who is Chief Scientist of this company. Frank Stenger has in the past written a number of joint papers with S. Johnson, D. Borup, J. Wiskin, M. Berggren and other present or past members of this group. Frank Stenger is also listed as an inventor on certain TechniScan patents.

*TechniScan, Inc.* recently won two awards for the development of a breast cancer scanner based on ultrasound inverse scattering tomography. It received the Stoel-Rives award for medical innovation. It also received the "Best of State Utah 2005" presentation for medical product development.

Currently, *TechniScan Inc.* has a prototype ultrasonic tomography machines installed and undergoing clinical evaluation the St. Mark's Hospital in Salt City, UT and one at McKay-Dee Hospital in Ogden, UT. These machines are the outgrowth of earlier research machines that used preliminary imaging algorithms based on use Sinc bases to carry out their inversion algorithms. This early work provides a beneficial balance of accuracy and speed and efficient use of computer memory since only 4 samples per wavelength are required using Sinc bases, whereas at least 8 samples per wavelength are required via other bases or sampling schemes.

Sinc basis sampling was featured in a *TechniScan, Inc.* publication outlining a method, using inverse scattering, that increased speed and accuracy for remote SONAR imaging of buried objects on the ocean bottom [129]

Other articles featuring Sinc methods of inverse problems and imaging include [130 , 131 , 132] .

We list here some inverse problems results of Frank Stenger which have yet been used commercially. The results listed here are based on the 3–d Helmholtz equation,

$$\nabla^2 u(\bar{r}) + k^2 (1 + f(\bar{r}))\, u(\bar{r}) = 0 \quad \text{in} \quad \mathcal{B}\,, \tag{25}$$

where $B$ is a bounded region in $\mathbb{R}^3$ , $f(\bar{r}) = c_0^2/c^2(\bar{r}) - 1$, with $c(\bar{r})$ and $c_0$ the speeds of sound in $\mathcal{B}$ and in $\mathbb{R} \setminus \mathcal{B}$ respectively, so that $f = 0$ on $\mathbb{R}^3\ \bar{\mathcal{B}}$, and with $k = \omega / c_0$ . A *source* with respect to the equation (25) is any function $v$ which satisfies in $\mathcal{B}$ the equation

$$\nabla^2 v(\bar{r}) + k^2\, v(\bar{r}) = 0\,. \tag{26}$$

It may be shown that the solution $u$ of (25) resulting from a source $v$ satisfying (26) satisfies the integral equation

$$u(\bar{r}) - v(\bar{r}) = k^2 \int\int\int_{\mathcal{B}} G(\bar{r}\,, \bar{r}'\,, k)\, f(\bar{r}')\, u(\bar{r}')\, d\bar{r}'\,, \tag{27}$$

where

$$G(\bar{r}\,, \bar{r}'\,, k) = \frac{\exp\left(i\,k\,|\bar{r} - \bar{r}'|\right)}{4\,\pi\,|\bar{r} - \bar{r}'|}\,. \tag{28}$$

For example, two easily produced sources in applications are a plane wave and a spherical source. These are given respectively by

$$\begin{array}{rcl} v(\bar{r}) & = & \exp(i\,\bar{k}\cdot\bar{r})\,, \quad |\bar{k}| = k \\ v(\bar{r}) & = & G(\bar{r}\,, \bar{r}_s\,, k)\,, \quad \bar{r}_s \notin \mathcal{B}\,. \end{array} \tag{29}$$

We also denote two points on the exterior of $\mathcal{B}$ : $\bar{r}_s$ – a source point, and $\bar{r}_d$ – a detector point.

Two types of approximations were popular in the past, since the solution of the Helmholtz equation is time consuming:

A. The *Born approximation:*

$$
\begin{aligned}
&u(\bar{r}) = e^{i\,\bar{k}\cdot\bar{r}}\,e^{W}\,, \\
&W = W_B(\bar{r}\,,\bar{r}_s\,,k) = \int\int\int_B G(\bar{r}\,,\bar{r}'\,,k)\,f(\bar{r}')\,e^{i\,\bar{k}\cdot\bar{r}}\,d\bar{r}\,.
\end{aligned} \tag{30}
$$

and

B. The *Rytov approximation*

$$
\begin{aligned}
&u(\bar{r}) = G(\bar{r}\,,\bar{r}_s\,,k)\,e^{W}\,, \\
&W = W_R(\bar{r}\,,\bar{r}_s\,,k) = \int\int\int_B G(\bar{r}\,,\bar{r}'\,,k)\,f(\bar{r})'\,G(\bar{r}'\,,\bar{r}_s\,,k)\,d\bar{r}\,,
\end{aligned} \tag{31}
$$

The following results in which $\alpha$ is a positive number depending on the smoothness of $f$ on the ray path connecting $\bar{r}_s$ and $\bar{r}_d$ were established in [60]:

$$
\begin{aligned}
&\frac{2\,i}{k}\,W_B(\bar{r}_d\,,\bar{r}_s\,,k) \\
&\quad = |\bar{r}_d - \bar{r}_s| \int_0^1 f((1-t)\,\bar{r}_d + t\bar{r}_s)\,dt + \mathcal{O}(k^{-\alpha})\,, \quad k \to \infty\,, \\
&\frac{4\,i}{k\,G(\bar{r}_d\,,\bar{r}_s\,,k)}\,W_R(\bar{r}_d\,,\bar{r}_s\,,k) \\
&\quad = |\bar{r}_d - \bar{r}_s| \int_0^1 f((1-t)\,\bar{r}_d + t\,\bar{r}_s)\,dt + \mathcal{O}(k^{-\alpha})\,, \quad k \to \infty\,.
\end{aligned} \tag{32}
$$

These results show that the ray paths for the Born and Rytov approximations are straight lines, and if the dominant terms of these approximations were known then one could use X-ray tomography algorithms to reconstruct the function $f$ in $\mathcal{B}$.

The *geometric optics* approximation to the solution $u$ of (25) satisfies the equation

$$
\frac{i}{k}\,\log\left(\frac{u(\bar{r}_s\,,k)}{u(\bar{r}_d\,,k)}\right) = \int_{\mathcal{P}} \sqrt{1 + f(\bar{r}(s))}\,ds + \mathcal{O}(k^{-\alpha})\,, \qquad k \to \infty\,, \tag{33}
$$

where $\mathcal{P}$ is the ray path (i.e., not a straight line) connecting $\bar{r}_s$ and $\bar{r}_d$.

Now, e.g., for the case of ultrasonic tomography involving frequencies $\nu$ from 2 to 4 megahertz, the difference between the actual values on the left hand sides of (32) and (33) and the corresponding terms on the right is very large, so that direct application of these asymptotic results is not very useful. Similarly, the left hand sides of each of the terms in (33) and (33) have a singularity at $k = \infty$, and so polynomial extrapolation to the limit via polynomials in $1/k$ does not work. On the other hand, it can be shown that the terms on the left hand sides of of (32) and (33) belong to the space $\mathbf{M}_{\alpha,d}(\varphi)$ as a function of $k$, (see §2.1, #5 above) with $\varphi(k) = \log(k - k_0)$, with $0 < k_0 < 2\pi\nu/c_0$, i.e., we can accurately approximate these functions with low degree rationals (see §2.1, #12 above). Additionally, when a transducer "fires", it contains all frequencies $\nu$ in an interval of frequencies, such as, e.g., $2 \times 10^6 < \nu < 4 \times 10^6$. We can thus use Thiele's method of extrapolation to the limit using a finite number of values of on $2\times 10^6/c_0 < k < 4\times 10^6/c_0$ to accurately predict the dominant terms on the right hand sides of (32) and (33). Noise is of course present in applications, and we must therefore first apply a noise reduction algorithm such as $\ell^1$ averaging [62] to remove the majority of the noise before applying Thiele's method.

2. The following result is established in [123]: *Given points $\bar{r}_0 \in \mathcal{B}$, $\bar{r} \in \mathbb{R}^3 \setminus \mathcal{B}$, and any $\varepsilon > 0$, there exist two sources $v_1$ and $v_2$ satisfying (26), such that $f(\bar{r}_0)$ may be computed to within an error of $\varepsilon$ using the values $u_1(\bar{r})$ and $u_2(\bar{r})$ by performing one addition, one multiplication, and one division.*

## 2.3 Miscellaneous Research Results of Frank Stenger.

Some of the more interesting "non-Sinc" research results of Frank Stenger are presented here.

1. In [19], a joint with P. Lipow (a former student of Schoenberg) it is shown that if $\{Q_n\}_1^\infty$ is any sequence of $n$–point quadrature formulas such that $Q_n(f)$ converges to $I(f) = \int_a^b f(x)\,dx$ as $n \to \infty$ for all continuous functions $f$ on a finite interval $[a,b]$, and given any strictly decreasing sequence of positive numbers $\{a_k\}_{k=1}^\infty$ there exists a function $f$ that is continuous on $[a,b]$, and a sub-sequence $\{Q_{n_k}\}_{k=1}^\infty$ such that $I(f) - Q_{n_k}(f) = a_k$, $k = 1, 2, \ldots,$ where $\|f\| = 3\,a_0$. That is, the sequence $\{Q_n\}$ can converge arbitrarily slowly. Such a result was previously known only for polynomial approxi-

mation of continuous functions. Moreover, it extends Polya's result on the non-convergence of the Newton–Cotes quadrature rules for all continuous functions.

2. In [32] Rosenberg and Stenger obtained an interesting result involving the "bisection" of triangles. Given a triangle $T_0$ with smallest interior angle $\alpha$, the process of bisection of this triangle is to draw a line segment from the mid-point of the longest edge to the opposite vertex (if there are two or three of the edges that are longest, then it is immaterial which longest edge is selected), thus producing two triangles. The result proved in [32] states that if $T$ is any member of the family of triangles produced by first bisecting $T_0$ to produce $T_1$ and $T_2$, then bisecting each of the two new triangles, and so on, then the size of the smallest interior angle of $T$ is at least $\alpha/2$. Although it was unknown to the authors of [32] when they wrote their paper, that the angles of triangles do not go to zero upon repeated bisection was a conjecture of finite element users, the result of which was required for convergence.

3. Also, during the same year, Stenger derived a novel formula via use of a combinatorial argument [34] for computing the topological degree of a continuous mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$, as a function of the signs of the coordinates of the map at a finite number of points on the boundary of the domain. The original proof published in [34] uses induction. A different proof for the two dimensional case is given on page 32 of Stenger text [87]. For example, if $B$ is a bounded planar region with boundary $\partial B$ containing the points $X_0 = (x_0, y_0)$, $X_1 = (x_1, y_1)$, $\ldots$, $X_N = (x_N, y_N)$, where these points are listed in counterclockwise fashion, and if at least one the two continuous components $(f, g)$ of $F$ is non-zero on each closed arc of $\partial B$ with end-points at $X_j$ and $X_{j+1}$, for $j = 0, 1, \ldots, N$, with $X_{N+1} = X_0$, then the topological degree of $F$ at $(0,0)$ relative to $B$ given by

$$d(F, B, (0,0)) = \frac{1}{8} \sum_{j=0}^{N} \begin{vmatrix} \operatorname{sgn} f(X_j) & \operatorname{sgn} g(X_j) \\ \operatorname{sgn} f(X_{j+1}) & \operatorname{sgn} g(X_{j+1}) \end{vmatrix} . \tag{34}$$

Here $\operatorname{sgn}(a)$ is defined for any real number $a$ by $\operatorname{sgn}(a) = 1$ if $a > 0$, $0$ if $a = 0$, and $-1$ if $a < 0$.

The above sum (34) is always an integer under the conditions of application of the formula, and as is well known, if $d(F, B, (0,0)) \neq 0$ then there are at least $|d(F, B, (0,0))|$ solutions of the equation $(f, g) = (0,0)$ in the interior of $B$.

4. In a recent colloquium talk in Math. at the Univ. of Utah, David Bailey from Lawrence Livermore Labs. stated that in computing $\pi$ to 20 billion places, his group used Sinc quadrature to check their results.

5. Asymptotic methods are, of course important, especially for deriving the dominant term of an expansion. It is also frequently possible to set up an indefinite integral, or a Volterra integral equation for either bounding the error of a truncated expansion, or for obtaining more terms of of an expansion. This integral, or Volterra integral equation can now be evaluated to uniform accuracy on the whole interval of interest via Sinc methods. For example, a derived asymptotic result (of an integral, a differential equation, etc.) might take the form

$$f(x) = g(x)\ (1 + \varepsilon(x))\ , \qquad x \in (a\,,\infty)\,,$$

with $g(x)$ explicitly known, followed by an estimate such as, $\varepsilon(x) = \mathcal{O}\left(x^{-c}\right)$, or a bound on $\varepsilon(x)$ which might depend on $x \in (a\,,\infty)$. Sinc methods enable a simple expression for $\varepsilon(x)$ that can be evaluated to arbitrary accuracy for any $x \in (a\,,\infty)$. See the IVP examples section of Sinc-Pack.

6. In [8] Stenger proved, among other things, if $Q_n(f)$ is the $n$–point Gaussian quadrature approximation to $I(f) = \int_{-1}^{1} f(x)\,dx$, if $f$ is analytic on the unit disc and integrable over $(-1,1)$, and if the even derivatives $f^{(2k)}(0)$ of $f$ are all of one sign, then $Q_n(f)$ converges monotonically to $I(f)$.

## 2.4 Program Packages of Frank Stenger.

The following program packages exist:

i. The quadrature package, *ALGORITHM 614. A FORTRAN Subroutine for Numerical Integration in* $\mathbf{H}^p$, written jointly with K. Sikorski and J. Schwing, in ACM TOMS 10 (1984) 152–160.

   This routine, originally written in FORTRAN, evaluates integrals in two ways: (a) To an arbitrary given accuracy of $\varepsilon$; or (b) Using a minimal number of points to achieve and accuracy, when $f/vp' \in \mathbf{M}_{\alpha,d}(\varphi)$, and we know both $\alpha$ and $d$. It is time to improve it, by rewriting it in Matlab, and by making it dependent on $\varphi$, thus shortening the program considerably.

ii. The ODE–IVP package [106], *ODE – IVP – PACK via Sinc Indefinite Integration and Newton's Method*, with SÅ. Gustafson, B. Keyes, M. O'Reilly, and K. Parker, published in "Numerical Algorithms" **20** (1999) 241–268 .

This package can be downloaded from Netlib. It is similar to Bill Gear's package for solving initial value problems of ordinary differential equations, except that it differs from his, or other packages, in the following ways:

- a. It uses Sinc indefinite integration, rather than step–by–step methods based on finite differences;
- b. The package yields arbitrary, uniform accuracy, for all problems, whereas other packages are accurate to within a least squares error;
- c. Other packages work only for finite intervals, whereas the present one yields solutions over arbitrary intervals, finite or infinite, and even contours;
- d. Classical methods suffer due to problems of instability, whereas this package does not; and
- e. Classical methods have trouble dealing with *Stiff problems*, whereas this package does not.

iii. The PDE package, *Ptolemy*, [107], prepared for his Ph.D. work, by *K. Parker*, in 1999.

It is written in Maple. It converts an elliptic PDE and IE (integral equations) over a curvilinear region to a system of algebraic equations, based on Sinc approximation of the derivatives of the PDE.

iv. *Handbook of Sinc Numerical Methods*, published by CRC Press (2010).

It consists of two parts:

(a) A *Tutorial of Sinc Methods*, a 470–page text, i containing novel derivations of the Sinc theory and one dimensional Sinc methods, via minimization of the use of complex variables, then derives novel methods of solution of PDE and integral equations via use of Sinc convolution and boundary integral methods. It is moreover shown in the tutorial that all solutions of linear PDE can be obtained via use of one dimensional Sinc methods, (i.e., via *separation of variables*– Stenger expends considerable effort to show that this is always possible) even over curvilinear regions, yielding solutions that are uniformly accurate and converge exponentially, as a function of the size of the one dimensional matrices. The multidimensional Laplace transforms of Green's functions are required for success of this endeavor, and to this end, Stenger was able to obtain the Laplace transforms of all of the free space Green's functions known to him. The resulting solution techniques are often orders of magnitude faster than current methods in use.

(b) A set of approximately 450 Matlab programs, written by Stenger. The use of some of these programs is illustrated in the above cited tutorial, for solving a variety of one dimensional and PDE problems. These are contained in a CD published with the text "Handbook of Sinc Numerical Methods", CRC Press (2010).

## 2.5 Textbooks.

The following textbooks are authored or co-authored by Stenger:

a. *Numerical Methods Based on Sinc and Analytic Functions,* approximately 565 pages, Computational Math. Series, Vol. 20, Springer–Verlag (1993).

b. *Selected Topics of Approximation and Computation*, with K. Sikorski and M. Kowalski, approximately 349 pages, Oxford University Press (1995). Was awarded "First Prize" by the Minister of Education in Poland, for the best research in 1995.

c. *Numerical Analysis,* textbook, with J. McNamee, 531 pages, in manuscript. McNamee has been deceased for over 12 years. The book was started by him and me about 20 years ago. During the past year I wrote three additional chapters, 2 on Sinc methods and one on asymptotic methods.

d. *Handbook of Sinc Numerical Methods*, CRC Press (2010). About 490 pages. Completed in 2010. this was already discussed above.

The following text is a nice introduction to Sinc methods:

e. *Sinc Methods for Quadrature and Differential Equations*, by J. Lund & K. Bowers, approximately 334 pages, SIAM (1992).

## Bibliography.


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