FoDA L20

Eigendecomposition + the Power Method

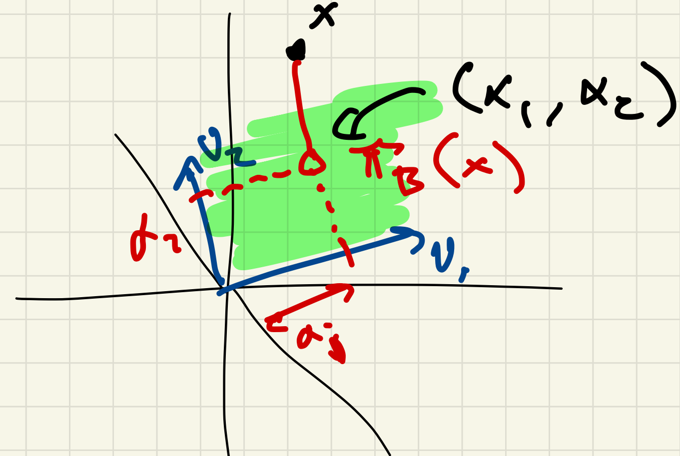
Data point $x \in \mathbb{R}^d$

$v_i \in \mathbb{R}^d$

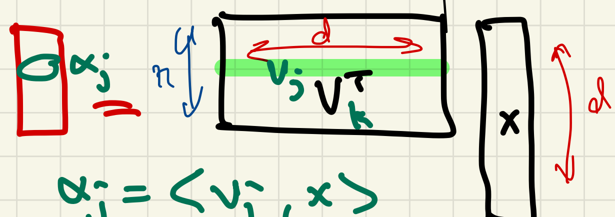
$\Pi_B(x)$ $\quad V_B = \{v_1, \dots v_k\}$

$$\Pi_B(x) = \sum_{j=1}^{k} v_j \underbrace{\langle v_j, x \rangle}_{\alpha_j}$$

$$= \sum_{j=1}^{k} v_j \, \alpha_j$$



$V^T x$



$\alpha_j = \langle v_j, x \rangle$

Eigenvalues & Eigenvectors

square matrix $M \in \mathbb{R}^{d \times d}$

eigenvalue

$$Mv = \lambda v$$

$v \in \mathbb{R}^d$

$\|v\| = 1$

eigenvector

M positive definite

"nice" case

d pairs

$v_1, v_2 \dots v_d$

$\lambda_1, \lambda_2, \dots \lambda_d$

$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_d > 0$

- $\|v_i\| = 1$
- $\langle v_i, v_j \rangle = 0 \quad i \neq j$

Input data $A \in \mathbb{R}^{n \times d}$

assume A full rank, $n > d$

$M = A^T A \in \mathbb{R}^{d \times d}$, rank d

$svd(A) = USV^T = A$

$MV = A^T A V = (VSU^T)(USV^T)V$

$= VSS = VS^2$ $\quad S^2 = diag(\sigma_1^2, \sigma_2^2, \ldots \sigma_d^2)$

$MV = VS^2$

$\forall j \quad Mv_j = v_j \sigma_j^2 \longrightarrow \lambda_j = \sigma_j^2$

eigen vectors, sing. vec

$v_j = v_j$

$M_L = A A^T \in \mathbb{R}^{n \times n}$

M_L eigenvectors $u_1, u_2, \dots u_n$ ← left sing. vectors of A

$\lambda_1, \lambda_2 \dots \lambda_d$

$\lambda_j \overset{\uparrow}{=} \sigma_j^2$ squared sing values

$\lambda_k \quad k = d+1 \dots n$

$\lambda_k = 0$

for $n > d$

A rank $= d$

Eigen decomposition

$$M = V L V^{-1}$$

orthogonal

$V = \{v_1, v_2, \dots v_d\}$

$L = \operatorname{diag}(\lambda_1, \lambda_2, \dots \lambda_d)$

$M = A^T A$

$V^{-1} = V^T$

$V^T V = I$

$$M^{-1} = (V L V^{-1})^{-1}$$

$$= V L^{-1} V^{-1} = V L^{-1} V^T$$

$$L^{-1} = \operatorname{diag}(1/\lambda_1, 1/\lambda_2, \dots, 1/\lambda_d)$$

Power Method

0. Initialize $u^{(0)}$ as random vector in $\mathbb{R}^d$

1. for $i = 1$ to q

$$u^{(i)} = M\, u^{(i-1)}$$

$$u^{(q)} = M \cdot M \cdots M\, u^{(0)} = M^q u^{(0)}$$

2. return $u^{(q)} / \| u^{(q)} \|$

be (approximately) the first eigen vector v_1 of M

$$\lambda_1 = \| M v_1 \|$$

Recovering all eigen values / vectors

v_1 orthogonal to $v_2 \dots v_d$

$M = A^T A$

(remove effect of v_1) $A_1 = A - A v_1 v_1^T$

$M_1 = A_1^T A_1$

$v_2 = \text{PowerMethod}(M_1, \varepsilon)$

$\lambda_2 = \| M_1 v_2 \|$

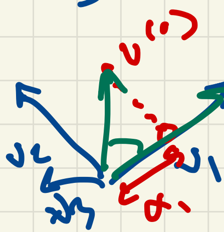
repeat d times

Why Power Method Works

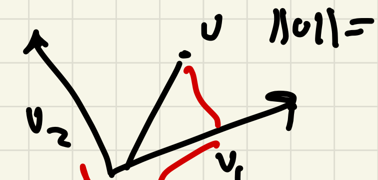
$$v_1 = M^{\ell} u^{(0)}$$

Say we know $M \to v_1, v_2, \dots v_d$ ← basis

$\lambda_1, \lambda_2, \dots \lambda_d$



$$u^{(0)} = \sum_{j=1}^{d} \alpha_j v_j \qquad \alpha_j = \langle u^{(0)}, v_j \rangle$$



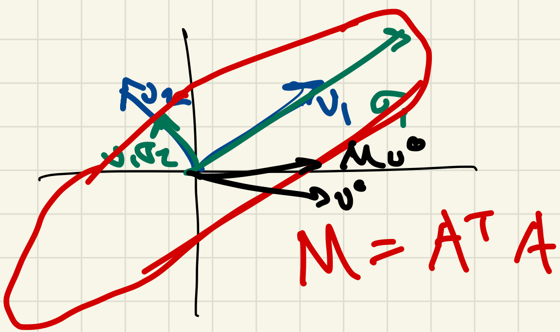
$$\alpha_1 \geq \frac{1}{2\sqrt{d}} \quad \text{w.p.} > \frac{1}{2}$$

$$1^2 = \sum_{j=1}^{d} \alpha_j^2$$

$$E[\alpha_j^2] = \frac{1}{d}$$

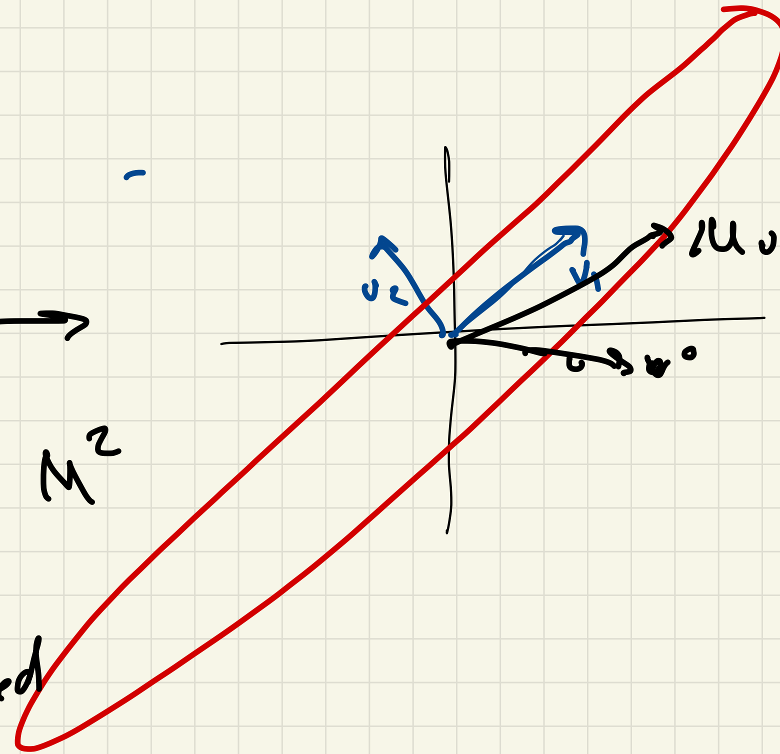
① Start $u^{(0)}$ not too far from v_1

② $M \to M^q$



$\longrightarrow$

M^2



Same basis
more stretched

$$M^q v_j = M \cdots M v_j = M^{(q-1)}(M v_j) = M^{(q-1)}(v_j \lambda_j)$$
$$= M^{(q-2)} v_j \lambda_j^2 = v_j \lambda_j^q$$

$$v = \frac{M^q u^{(0)}}{\|M^q u^{(0)}\|} = \frac{\sum_{j=1}^{d} \alpha_j v_j \lambda_j^q}{\sqrt{\sum_{j=1}^{d} (\alpha_j \lambda_j^q)^2}} \in \mathbb{R}^d$$

$$v = M^q u^{(0)}$$

$$|\langle v, v_1 \rangle| = \frac{\alpha_1 \lambda_1^q}{\sqrt{\sum_{j=1}^{d} (\alpha_j \lambda_j^q)^2}}$$

$$\geq \frac{\alpha_1 \lambda_1^q}{\sqrt{\alpha_1^2 \lambda_1^{2q} + d \lambda_2^{2q}}} \geq \frac{\alpha_1 \lambda_1^q}{\alpha_1 \lambda_1^q + \lambda_2^q \sqrt{d}} = 1 - \frac{\lambda_2^q \sqrt{d}}{\alpha_1 \lambda_1^q + \cancel{\lambda_2^q \sqrt{d}}}$$

$$\geq \boxed{1 - 2d\left(\frac{\lambda_2}{\lambda_1}\right)^q}$$

$$\alpha_1 \geq \frac{1}{2\sqrt{d}}$$

← convergence depends on gap/ratio $\frac{\lambda_2}{\lambda_1}$